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Some inequalities for operator monotone functions

Silvestru Sever Dragomir

Abstract In this paper we show that, if that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ then there exist $b\geq 0$ and a positive measure m on $[0,\infty)$ such that

$$\begin{aligned}&\left[f\left(B\right)-f\left(A\right)\right]\left(B-A\right)=\\&\quad=b\left(B-A\right)^{2}+\int_{0}^{\infty}s^{2}\left[\int_{0}^{1}\left[\left(\left(1-t\right)A+tB+s\right)^{-1}\left(B-A\right)\right]^{2}dt\right]dm\left(s\right)\end{aligned}$$

for all A, $B>0$. Some necessary and sufficient conditions for the operators A, $B>0$ such that the inequality

$$f\left(B\right)B+f\left(A\right)A\geq f\left(A\right)B+f\left(B\right)A$$

holds for any operator monotone function f on $[0,\infty)$ are also given.

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1 Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. We say that the operators A, $B:H\to H$ satisfy the relation $A\geq B$ if $A-B\geq 0$. A real valued continuous function $f(t)$ on $(0,\infty)$ is said to be *operator monotone* if $f(A)\geq f(B)$ holds for any $A\geq B>0$. By $f(T)$ we denote the operator that can be defined by the use of the continuous functional calculus of selfadjoint operators T in Hilbert spaces.

In 1934, K. Löwner [9] had given a definitive characterization of operator monotone functions as follows, see for instance [2, p. 144-145]:

Theorem 1.1. *A function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{ts}{t+s}dm(s) \tag{1.1}$$

where $a\in\mathbb{R}$ and $b\geq 0$ and a positive measure m on $[0,\infty)$ such that

$$\int_0^\infty \frac{s}{1+s}dm(s)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:[0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [8]. As a consequence, we have the Löwner-Heinz operator inequality $A^\alpha\geq B^\alpha\geq 0$ provided that $A\geq B\geq 0$ and $\alpha\in[0,1]$. It is also well known that the logarithmic function ln is operator monotone on $(0,\infty)$.

For several examples of operator monotone functions, see [5], [10], [4] and the references therein. For recent operator inequalities related to operator monotone functions, see [1], [11] and [12].

In this paper we show that, if that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ then there exist $b\geq 0$ and a positive measure m on $(0,\infty)$ such that

$$\begin{aligned}&[f(B)-f(A)](B-A)=\\&=b(B-A)^2+\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2dt\right]dm(s)\end{aligned}$$

for all $A,\ B>0$. Some necessary and sufficient conditions for the operators $A,\ B>0$ such that the inequality

$$f(B)B+f(A)A\geq f(A)B+f(B)A$$

holds for any operator monotone function f on $[0,\infty)$ are also given.

2 Main Results

We have the following identities of interest:

Theorem 2.1. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.1). Then for all $A,\ B>0$ we have*

$$\begin{aligned}&[f(B)-f(A)](B-A)=\\&=b(B-A)^2+\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2dt\right]dm(s)\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&(B-A)[f(B)-f(A)]=\\&=b(B-A)^2+\int_0^\infty s^2\left[\int_0^1\left[(B-A)((1-t)A+tB+s)^{-1}\right]^2dt\right]dm(s).\end{aligned} \tag{2.2}$$

Proof. Since the function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$, then f can be written as in the equation (1.1) and for A, $B > 0$ we have the representation

$$f(B) - f(A) = b(B - A) + \int_0^\infty s\left[B(B+s)^{-1} - A(A+s)^{-1}\right] dm(s). \tag{2.3}$$

Observe that for $s > 0$

$$\begin{aligned} B(B+s)^{-1} - A(A+s)^{-1} = \\ &= (B+s-s)(B+s)^{-1} - (A+s-s)(A+s)^{-1} \\ &= (B+s)(B+s)^{-1} - s(B+s)^{-1} - (A+s)(A+s)^{-1} + s(A+s)^{-1} \\ &= 1 - s(B+s)^{-1} - 1 + s(A+s1_H)^{-1} \\ &= s\left[(A+s)^{-1} - (B+s)^{-1}\right]. \end{aligned}$$

Therefore, (2.3) becomes, see also [6]

$$f(B) - f(A) = b(B - A) + \int_0^\infty s^2\left[(A+s)^{-1} - (B+s)^{-1}\right] dm(s). \tag{2.4}$$

Let T, $S > 0$. The function $f(t) = -t^{-1}$ is operator monotonic on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t \to 0}\left[\frac{f(T + tS) - f(T)}{t}\right] = T^{-1}ST^{-1} \tag{2.5}$$

for T, $S > 0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0, 1]$ by

$$f_{C,D}(t) = f((1-t)C + tD), \ t \in [0, 1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C, D] := \{(1-t)C + tD, \ t \in [0, 1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\, dt = \int_0^1 \nabla f_{(1-t)C+tD}(D - C)\, dt. \tag{2.6}$$

If we write this equality for the function $f(t) = -t^{-1}$ and C, $D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1}(D - C)((1-t)C + tD)^{-1}\, dt. \tag{2.7}$$

Now, if we replace in (2.7) $C = A + s1_H$ and $D = B + s1_H$ for $s > 0$, then

$$(A+s)^{-1} - (B+s)^{-1} = \int_0^1 ((1-t)A + tB + s)^{-1}(B - A)((1-t)A + tB + s)^{-1}\, dt. \tag{2.8}$$

By the representation (2.4), we derive the following identity of interest

$$\begin{aligned} f(B)-f(A) = & b(B-A)+\int_0^\infty s^2\left[\int_0^1 ((1-t)A+tB+s)^{-1}\right. \\ & \left.\times (B-A)((1-t)A+tB+s)^{-1}dt\right]dm(s) \end{aligned} \tag{2.9}$$

for A, $B>0$.

If we multiply this identity at the right with $B-A$ we get

$$\begin{aligned} (f(B)-f(A))(B-A) = & b(B-A)^2+\int_0^\infty s^2\left[\int_0^1 ((1-t)A+tB+s)^{-1}\right. \\ & \left.\times (B-A)((1-t)A+tB+s)^{-1}(B-A)dt\right]dm(s) \\ = & \int_0^\infty s^2\left[\int_0^1 \left[((1-t)A+tB+s)^{-1}(B-A)\right]^2 dt\right]dm(s) \end{aligned} \tag{2.10}$$

for A, $B>0$ and the equality (2.1) is proved.

The equality (2.2) follows by multiplying (2.9) at the left. □

In the following, in order to simplify terminology, when we write $T\geq 0$, then we automatically assume that the operator T is selfadjoint.

In the note [3], Fujii and Nakamoto showed that the inequality

$$(f(B)-f(A))(B-A)\geq 0$$

does not hold in general for A, $B>0$.

They also proved the following interesting inequality:

Proposition 2.2 ([3, Proposition 2]). *If C, $D>0$ and $CD^{-1}+DC^{-1}$ is selfadjoint, then*

$$CD^{-1}+DC^{-1}\geq 2. \tag{2.11}$$

Proof. Indeed, as shown in [3], if we put $T=CD^{-1}$, then $V=T+T^{-1}$ is selfadjoint by the assumption. Note that the spectrum $\mathrm{Sp}(T)$ of T is included in $(0,\infty)$, because C, $D>0$ and $\mathrm{Sp}(T)=\mathrm{Sp}\left(C^{1/2}D^{-1}C^{1/2}\right)$. Since $\mathrm{Sp}(V)=\left\{t+\frac{1}{t},\ t\in\mathrm{Sp}(T)\right\}$ by the spectral mapping theorem for rational functions, hence we have $T+T^{-1}\geq 2$. □

As a consequence, they obtained the following result:

Theorem 2.3 ([3, Theorem 6]). *If*

(i') Operator $A(B+s)^{-1}+B(A+s)^{-1}$ is selfadjoint for all $s\geq 0$,

then $(B-A)(f(B)-f(A))\geq 0$.

Some necessary and sufficient conditions for the operators A, $B>0$ such that the inequality $(f(B)-f(A))(B-A)\geq 0$ holds for any operator monotone function f on $[0,\infty)$ are included in the following theorem.

Theorem 2.4. *Let* A, $B > 0$. *The following statements are equivalent:*

(i) For all $s \geq 0$,
$$\left(A+s\right)^{-1}\left(B+s\right)+\left(B+s\right)^{-1}\left(A+s\right)\geq 2. \tag{2.12}$$

(ii) For all $s \geq 0$,
$$\int_0^1 \left[\left(\left(1-t\right)A+tB+s\right)^{-1}\left(B-A\right)\right]^2 dt \geq 0.$$

(iii) For all $s \geq 0$,
$$\left(\ell_s\left(B\right)-\ell_s\left(A\right)\right)\left(B-A\right)\geq 0,$$
where $\ell_s\left(t\right)=-(t+s)^{-1}$, $t>0$.

(iv) For all operator monotone function f *on* $[0,\infty)$,
$$\left(f\left(B\right)-f\left(A\right)\right)\left(B-A\right)\geq 0. \tag{2.13}$$

(v) For all operator monotone function f *on* $[0,\infty)$,
$$\left(B-A\right)\left(f\left(B\right)-f\left(A\right)\right)\geq 0. \tag{2.14}$$

Proof. From (2.8) we have, by multiplying at right with $B-A$ that
$$\begin{aligned}\left[\left(A+s\right)^{-1}-\left(B+s\right)^{-1}\right]\left(B-A\right)=&\\ &=\int_0^1\left(\left(1-t\right)A+tB+s\right)^{-1}\left(B-A\right)\left(\left(1-t\right)A+tB+s\right)^{-1}\left(B-A\right)dt\\ &=\int_0^1\left[\left(\left(1-t\right)A+tB+s\right)^{-1}\left(B-A\right)\right]^2 dt\end{aligned}$$
for all $s \geq 0$.

Also,
$$\begin{aligned}\left[\left(A+s\right)^{-1}-\left(B+s\right)^{-1}\right]\left(B-A\right)&=\left[\left(A+s\right)^{-1}-\left(B+s\right)^{-1}\right]\left[B+s-\left(A+s\right)\right]\\ &=\left(A+s\right)^{-1}\left(B+s\right)+\left(B+s\right)^{-1}\left(A+s\right)-2\end{aligned}$$
for all $s \geq 0$.

Therefore
$$\begin{aligned}\left(\ell_s\left(B\right)-\ell_s\left(A\right)\right)\left(B-A\right)&=\left(A+s\right)^{-1}\left(B+s\right)+\left(B+s\right)^{-1}\left(A+s\right)-2\\ &=\int_0^1\left[\left(\left(1-t\right)A+tB+s\right)^{-1}\left(B-A\right)\right]^2 dt\end{aligned} \tag{2.15}$$
for all $s \geq 0$.

The identity (2.15) reveals that the statements (i), (ii) and (iii) are equivalent.

Since for fixed $s \geq 0$, $\ell_s\left(t\right)$ is operator monotone function on $\left(0,\infty\right)$, then statement (iv) implies (iii).

Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. Then for all A, $B>0$ we have

$$\begin{aligned}
&[f(B)-f(A)](B-A)= \\
&\qquad =b(B-A)^2+\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2dt\right]dm(s) \\
&\qquad \geq\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2dt\right]dm(s)
\end{aligned} \tag{2.16}$$

where $b\geq 0$ and m is a positive measure on $[0,\infty)$.

If (ii) is valid, then

$$\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2dt\right]dm(s)\geq 0$$

and by (2.16) we obtain (2.13).

Define the operator $K:=(f(B)-f(A))(B-A)$. Since

$$\begin{aligned}
K^* &= [(f(B)-f(A))(B-A)]^*=(B-A)^*(f(B)-f(A))^* \\
&= (B-A)(f(B)-f(A))
\end{aligned}$$

then the fact that K is selfadjoint is equivalent to

$$(f(B)-f(A))(B-A)=(B-A)(f(B)-f(A)),$$

which is also equivalent to the fact that

$$f(A)B+f(B)A=Bf(A)+Af(B).$$

These prove the equivalence between (iv) and (v). $\square$

Remark 2.1. The identity

$$(B-A)(\ell_s(B)-\ell_s(A))=(B+s)(A+s)^{-1}+(A+s)(B+s)^{-1}-2$$

for $s\geq 0$ was the main tool in the proof of Theorem 6, [3].

We can state:

Corollary 2.5. *Let A, $B>0$. The statement (i) is equivalent to the inequality*

$$f(B)B+f(A)A\geq f(A)B+f(B)A, \tag{2.17}$$

for all f an operator monotone function on $(0,\infty)$.

Observe that, in fact we have:

Proposition 2.6. *Let A, $B>0$, then the statements (i) and (i') are equivalent.*

Proof. Notice that for all $s \geq 0$,

$$\begin{aligned}(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s) &= \\ &= (A+s)^{-1}B+(B+s)^{-1}A+s(A+s)^{-1}+s(B+s)^{-1}\end{aligned} \tag{2.18}$$

Also, the operator $s(A+s)^{-1}+s(B+s)^{-1}$ is selfadjoint for $s \geq 0$.

If the statement *(i)* holds, then $(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)$ is selfadjoint and by (2.18) we must have that $(A+s)^{-1}B+(B+s)^{-1}A$ is selfadjoint, which shows that

$$\left((A+s)^{-1}B+(B+s)^{-1}A\right)^{*}=B(A+s)^{-1}+A(B+s)^{-1}$$

is selfadjoint, namely *(i')* is true.

If the statement *(i')* holds, then by (2.18) we get

$$(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)$$

is selfadjoint and by (2.11) for $C=(A+s)^{-1}$, $D=(B+s)^{-1}$ we obtain the inequality (2.12), namely *(i)* is true. □

We define the class of operators

$$\mathfrak{Cl}_{(0,\infty)}(H) := \{(A,B) \mid A,\ B>0 \text{ and satisfy condition } (i')\}.$$

We observe that if $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$ then $(B,A) \in \mathfrak{Cl}_{(0,\infty)}(H)$.

Also if $AB=BA$, $A,B>0$, then $U_s := (A+s)^{-1}(B+s)$ and $U_s^{-1}=(B+s)^{-1}(A+s)$ are selfadjoint and since $U_s+U_s^{-1} \geq 2$, $s \geq 0$ we derive that $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$. Therefore, if $\mathfrak{Co}_{(0,\infty)}(H)$ is the class of all pairs of commutative operators A, $B>0$, then we have

$$\emptyset \neq \mathfrak{Co}_{(0,\infty)}(H) \subset \mathfrak{Cl}_{(0,\infty)}(H). \tag{2.19}$$

Corollary 2.7. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. If $g:I\to[0,\infty)$ is continuous, then for all selfadjoint operators A, B with spectra in I for which $(g(A),g(B)) \in \mathfrak{Cl}_{(0,\infty)}(H)$ we have*

$$(f\circ g)(B)\,g(B)+(f\circ g)(A)\,g(A) \geq (f\circ g)(A)\,g(B)+(f\circ g)(B)\,g(A). \tag{2.20}$$

Follows by Theorem 2.1 by replacing A with $g(A)$, B with $g(B)$ and using the composition rule for continuous functions of selfadjoint operators which gives that $f(g(A))=(f\circ g)(A)$ and $f(g(B))=(f\circ g)(B)$, see for instance [7, p. 49].

Corollary 2.8. *Assume that the function $f:[0,\infty)\to[0,\infty)$ is operator monotone in $[0,\infty)$. Then*

$$B^2(f(B))^{-1}+A^2(f(A))^{-1} \geq (f(A))^{-1}AB+(f(B))^{-1}BA, \tag{2.21}$$

for all $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$.

Follows by Corollary 2.5 and the fact that $t/f(t)$ is also operator monotone on $(0,\infty)$.

Remark 2.2. Since $f:[0,\infty)\to\mathbb{R}$, $f(t)=t^{\alpha}$ is an operator monotone function for any $\alpha\in[0,1]$, then by (2.17) we get

$$B^{\alpha+1}+A^{\alpha+1}\geq A^{\alpha}B+B^{\alpha}A, \tag{2.22}$$

for all $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$.

Suppose that $p>1$. Then by taking $f(t)=t^{1/p}$ and $g(t)=t^{p}$, for $t\in(0,\infty)$, in Corollary 2.7, we get

$$B^{p+1}+A^{p+1}\geq AB^{p}+BA^{p},$$

for all $A,\ B>0$ such that $(A^{p},B^{p})\in\mathfrak{Cl}_{(0,\infty)}(H)$.

3 Applications for Integral and Discrete Inequalities

We have the following integral inequality:

Proposition 3.1. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. Then for all $A,\ B>0$ with $\left((1-t)A+tB,\frac{A+B}{2}\right)\in\mathfrak{Cl}_{(0,\infty)}(H)$ for all $t\in[0,1]$,*

$$\int_0^1((1-t)A+tB)f((1-t)A+tB)\,dt\geq\left(\int_0^1 f((1-t)A+tB)\,dt\right)\frac{A+B}{2}. \tag{3.1}$$

Proof. From (2.17) we get

$$\begin{aligned}((1-t)A+tB)f((1-t)A+tB)+\frac{A+B}{2}f\left(\frac{A+B}{2}\right)\geq\\ \geq f\left(\frac{A+B}{2}\right)((1-t)A+tB)+f((1-t)A+tB)\frac{A+B}{2}\end{aligned} \tag{3.2}$$

for all $t\in[0,1]$.

By taking the integral in (3.2) we get

$$\begin{aligned}\int_0^1((1-t)A+tB)f((1-t)A+tB)\,dt+\frac{A+B}{2}f\left(\frac{A+B}{2}\right)\geq\\ \geq f\left(\frac{A+B}{2}\right)\int_0^1((1-t)A+tB)\,dt+\left(\int_0^1 f((1-t)A+tB)\,dt\right)\frac{A+B}{2}\end{aligned} \tag{3.3}$$

and since

$$\int_0^1((1-t)A+tB)\,dt=\frac{A+B}{2},$$

hence by (3.3) we derive

$$\begin{aligned}\int_0^1((1-t)A+tB)f((1-t)A+tB)\,dt+\frac{A+B}{2}f\left(\frac{A+B}{2}\right)\geq\\ \geq f\left(\frac{A+B}{2}\right)\frac{A+B}{2}+\left(\int_0^1 f((1-t)A+tB)\,dt\right)\frac{A+B}{2},\end{aligned}$$

which is equivalent to the first inequality in (3.1).

The second inequality in (3.1) follows by the second part of (2.17). □

If we take $f(t) = t^r$, $r \in (0,1)$ in (3.1), then we get

$$\int_0^1 ((1-t)A + tB)^{r+1} dt \geq \left(\int_0^1 ((1-t)A + tB)^r dt \right) \frac{A+B}{2}, \tag{3.4}$$

for all A, $B > 0$ with $\left((1-t)A + tB, \frac{A+B}{2}\right) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for all $t \in [0,1]$.

We have the following Chebychev type operator inequality:

Proposition 3.2. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$. If $g : I \to [0,\infty)$ is continuous, then for all selfadjoint operators A_k, $k = 1,...,n$ with spectra in I and such that $(g(A_k), g(A_j)) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for $j,k = 1,...,n$ and $p_k \geq 0$, $k = 1,...,n$ with $\sum_{k=1}^n p_k = 1$,*

$$\sum_{k=1}^n p_k (f \circ g)(A_k) g(A_k) \geq \sum_{k=1}^n p_k (f \circ g)(A_k) \sum_{k=1}^n p_k g(A_k). \tag{3.5}$$

Proof. From (2.20) we get

$$(f \circ g)(A_k) g(A_k) + (f \circ g)(A_j) g(A_j) \geq (f \circ g)(A_j) g(A_k) + (f \circ g)(A_k) g(A_j) \tag{3.6}$$

for all k, $j \in \{1,...,n\}$.

If we multiply (3.6) by $p_k p_j \geq 0$ and sum over k and j from 1 to n, we get

$$\sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_k) g(A_k) + \sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_j) g(A_j) \geq$$
$$\geq \sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_j) g(A_k) + \sum_{k=1}^n \sum_{j=1}^n p_k p_j (f \circ g)(A_k) g(A_j),$$

which is equivalent to

$$\sum_{k=1}^n p_k (f \circ g)(A_k) g(A_k) + \sum_{j=1}^n p_j (f \circ g)(A_j) g(A_j) \geq$$
$$\geq \sum_{j=1}^n p_j (f \circ g)(A_j) \sum_{k=1}^n p_k g(A_k) + \sum_{k=1}^n p_k (f \circ g)(A_k) \sum_{j=1}^n p_j g(A_j)$$

that is equivalent to the first part of (3.5). □

Remark 3.1. If the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and $A_k > 0$, $k = 1,...,n$, with $(A_k, A_j) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for j, $k = 1,...,n$ and $p_k \geq 0$, $k = 1,...,n$ with $\sum_{k=1}^n p_k = 1$, then by (3.5),

$$\sum_{k=1}^n p_k A_k f(A_k) \geq \sum_{k=1}^n p_k f(A_k) \sum_{k=1}^n p_k A_k. \tag{3.7}$$

In particular, for $f(t) = t^r$, $r \in (0,1)$ we get

$$\sum_{k=1}^n p_k A_k^{r+1} \geq \sum_{k=1}^n p_k A_k^r \sum_{k=1}^n p_k A_k. \tag{3.8}$$

4 Conclusion

In this paper we obtained an integral representation for the operator

$$[f(B) - f(A)](B - A) \text{ for } A,\ B > 0$$

and f an operator monotone function on $(0, \infty)$ and provided some necessary and sufficient conditions for this product to be positive in the operator order. As applications, we obtained some Chebychev type integral and discrete operator inequalities with examples for powers. The obtained result can be extended for an operator monotone function f and $(A_t)_{t \in T}$ a bounded continuous field of positive operators in $\mathcal{B}(H)$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ that satisfy certain conditions while $(p_t)_{t \in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$.

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Silvestru Sever Dragomir

Mathematics, College of Engineering & Science,
Victoria University, PO Box 14428,
Melbourne City, MC 8001, Australia.
and
DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences,
School of Computer Science & Applied Mathematics,
University of the Witwatersrand, Johannesburg, South Africa.

E-mail: `sever.dragomir@vu.edu.au`