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A refinement of Grüss inequality for the complex integral

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A refinement of Grüss inequality for the complex integral [1]

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Abstract

Assume that f and g are continuous on γ, $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and the *complex Čebyšev functional* is defined by

$$\mathcal{D}_\gamma(f, g) := \frac{1}{w - u} \int_\gamma f(z) g(z) \, dz - \frac{1}{w - u} \int_\gamma f(z) \, dz \frac{1}{w - u} \int_\gamma g(z) \, dz.$$

In this paper we establish some Grüss type inequalities for $\mathcal{D}_\gamma(f, g)$ under some complex boundedness conditions for the functions f and g.

1 Introduction

For two Lebesgue integrable functions f, $g : [a, b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$C(f, g) := \frac{1}{b - a} \int_a^b f(t) g(t) \, dt - \frac{1}{b - a} \int_a^b f(t) \, dt \frac{1}{b - a} \int_a^b g(t) \, dt.$$

In 1934, G. Grüss [17] showed that

$$(1) \qquad |C(f,g)| \leq \frac{1}{4}(M-m)(N-n),$$

provided m, M, n, N are real numbers with the property that

$$(2) \qquad -\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a,b].$$

The constant $\frac{1}{4}$ is best possible in (1) in the sense that it cannot be replaced by a smaller one.

In [6], P. Cerone and S. S. Dragomir proved the following inequalities:

$$(3) \qquad |C(f,g)| \leq \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt, \\ \\ \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right|^p dt \right)^{\frac{1}{p}}, \\ \qquad\qquad\qquad\qquad \text{where } p > 1, \ 1/p + 1/q = 1. \end{cases}$$

For $\gamma = 0$, we get from the first inequality in (3)

$$(4) \qquad |C(f,g)| \leq \|g\|_\infty \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (3) we can deduce the following result obtained by Cheng and Sun [9]

$$(5) \qquad |C(f,g)| \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\, ds \right| dt.$$

The constant $\frac{1}{2}$ is best in (5) as shown by Cerone and Dragomir in [7].

For other inequality of Grüss' type see [1]-[5], [7]-[16], [18]-[23] and [25]-[28].

In order to extend Grüss' inequality to complex integral we need the following preparations.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t)) z'(t)\, dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\, dz := \int_{\gamma_{u,v}} f(z)\, dz + \int_{\gamma_{v,w}} f(z)\, dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\left|z'(t)\right|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b \left|z'(t)\right|\,dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$(6) \qquad \int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz.$$

We recall also the *triangle inequality* for the complex integral, namely

$$(7) \qquad \left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma)$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma}|f(z)|$.

We also define the *p*-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If $p,\,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $f : \gamma \to \mathbb{C}$ a continuous function on γ. Define the quantity:

$$(8) \qquad \mathcal{P}_\gamma\left(f,\overline{f}\right) = \frac{1}{\ell(\gamma)}\int_\gamma |f(z)|^2\,|dz| - \left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2$$

$$= \frac{1}{\ell(\gamma)}\int_\gamma \left|f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2\,|dv| \ge 0.$$

If f and g are continuous on γ, we consider the *complex Čebyšev functional* defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz\,\frac{1}{w-u}\int_\gamma g(z)\,dz.$$

In this paper we establish some Grüss type inequalities for $\mathcal{D}_\gamma(f,g)$ under some complex boundedness conditions for the functions f and g.

2   Some Preliminary Results

We have the following equalities:

Lemma 1 *Assume that the path γ is piecewise smooth and $f : \gamma \to \mathbb{C}$ is continuous on γ. Then for $\phi, \Phi \in \mathbb{C}$ with $\phi \neq \Phi$ we have*

$$(9) \quad \mathcal{P}_\gamma\left(f, \overline{f}\right) = \mathrm{Re}\left[\left(\Phi - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right)\left(\frac{1}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz| - \overline{\phi}\right)\right]$$
$$- \frac{1}{\ell(\gamma)}\int_\gamma \mathrm{Re}\left[(\Phi - f(z))\left(\overline{f(z)} - \overline{\phi}\right)\right]|dz|$$

and, equivalently,

$$(10) \quad \mathcal{P}_\gamma\left(f, \overline{f}\right) = \frac{1}{4}\left|\Phi - \phi\right|^2 - \left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz| - \frac{\phi + \Phi}{2}\right|^2$$
$$- \frac{1}{\ell(\gamma)}\int_\gamma \mathrm{Re}\left[(\Phi - f(z))\left(\overline{f(z)} - \overline{\phi}\right)\right]|dz|.$$

Proof. We have

$$I_1 := \mathrm{Re}\left[\left(\Phi - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right)\left(\frac{1}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz| - \overline{\phi}\right)\right]$$
$$= \mathrm{Re}\left[\frac{\Phi}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz| + \frac{\overline{\phi}}{\ell(\gamma)}\int_\gamma f(z)\,|dz| - \Phi\overline{\phi} - \left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2\right]$$
$$= \mathrm{Re}\left(\frac{\Phi}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz|\right) + \mathrm{Re}\left(\frac{\overline{\phi}}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right)$$
$$- \mathrm{Re}\left(\Phi\overline{\phi}\right) - \left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2$$

and

$$I_2 := \frac{1}{\ell(\gamma)}\int_\gamma \mathrm{Re}\left[(\Phi - f(z))\left(\overline{f(z)} - \overline{\phi}\right)\right]|dz|$$
$$= \frac{1}{\ell(\gamma)}\int_\gamma \mathrm{Re}\left[\Phi\overline{f(z)} + \overline{\phi}f(z) - \Phi\overline{\phi} - |f(z)|^2\right]|dz|$$
$$= \frac{1}{\ell(\gamma)}\int_\gamma \left[\mathrm{Re}\left(\Phi\overline{f(z)}\right) + \mathrm{Re}\left(\overline{\phi}f(z)\right) - \mathrm{Re}\left(\Phi\overline{\phi}\right) - |f(z)|^2\right]|dz|$$
$$= \mathrm{Re}\left(\frac{\Phi}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz|\right) + \mathrm{Re}\left(\frac{\overline{\phi}}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right)$$
$$- \mathrm{Re}\left(\Phi\overline{\phi}\right) - \frac{1}{\ell(\gamma)}\int_\gamma |f(z)|^2\,|dz|.$$

Therefore

$$I_1 - I_2 = \frac{1}{\ell(\gamma)} \int_\gamma |f(z)|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right|^2$$

proving the identity (9).

We have the equality for complex numbers

$$\text{Re}\left[(\Phi - w)(\overline{w} - \overline{\phi}) \right] = \frac{1}{4} |\Phi - \phi|^2 - \left| w - \frac{\phi + \Phi}{2} \right|^2$$

for $w \in \mathbb{C}$, then by taking

$$w = \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz|$$

in this equality, we get

$$\text{Re}\left[\left(\Phi - \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| \right) \left(\frac{1}{\ell(\gamma)} \int_\gamma \overline{f(z)} |dz| - \overline{\phi} \right) \right]$$
$$= \frac{1}{4} |\Phi - \phi|^2 - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right|^2$$

and by (9) we obtain the desired result (10).

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. Now, for ϕ, $\Phi \in \mathbb{C}$, define the sets of complex-valued functions

$$\bar{U}_\gamma(\phi, \Phi) := \left\{ f : \gamma \to \mathbb{C} \mid \text{Re}\left[(\Phi - f(z)) \left(\overline{f(z)} - \overline{\phi} \right) \right] \geq 0 \text{ for each } z \in \gamma \right\}$$

and

$$\bar{\Delta}_\gamma(\phi, \Phi) := \left\{ f : \gamma \to \mathbb{C} \mid \left| f(z) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \text{ for each } z \in \gamma \right\}.$$

The following representation result may be stated.

Proposition 2 *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_\gamma(\phi, \Phi)$ and $\bar{\Delta}_\gamma(\phi, \Phi)$ are nonempty, convex and closed sets and*

(11) $$\bar{U}_\gamma(\phi, \Phi) = \bar{\Delta}_\gamma(\phi, \Phi).$$

Proof. We observe that for any $w \in \mathbb{C}$ we have the equivalence

$$\left| w - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\text{Re}\left[(\Phi - w)(\overline{w} - \overline{\phi}) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}\left|\Phi-\phi\right|^2-\left|w-\frac{\phi+\Phi}{2}\right|^2=\operatorname{Re}\left[\left(\Phi-w\right)\left(\overline{w}-\overline{\phi}\right)\right]$$

that holds for any $w\in\mathbb{C}$.

The equality (11) is thus a simple consequence of this fact.

On making use of the complex numbers field properties we can also state that:

Corollary 3 *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$,we have that*

$$(12)\qquad \bar{U}_\gamma\left(\phi,\Phi\right)=\{f:\gamma\to\mathbb{C}\mid\left(\operatorname{Re}\Phi-\operatorname{Re}f\left(z\right)\right)\left(\operatorname{Re}f\left(z\right)-\operatorname{Re}\phi\right)$$
$$+\left(\operatorname{Im}\Phi-\operatorname{Im}f\left(z\right)\right)\left(\operatorname{Im}f\left(z\right)-\operatorname{Im}\phi\right)\geq 0\text{ }for\text{ }each\text{ }z\in\gamma\}.$$

Now, if we assume that $\operatorname{Re}\left(\Phi\right)\geq\operatorname{Re}\left(\phi\right)$ and $\operatorname{Im}\left(\Phi\right)\geq\operatorname{Im}\left(\phi\right)$, then we can define the following set of functions as well:

$$(13)\quad \bar{S}_\gamma\left(\phi,\Phi\right):=\{f:\gamma\to\mathbb{C}\mid\operatorname{Re}\left(\Phi\right)\geq\operatorname{Re}f\left(z\right)\geq\operatorname{Re}\left(\phi\right)$$
$$\text{and }\operatorname{Im}\left(\Phi\right)\geq\operatorname{Im}f\left(z\right)\geq\operatorname{Im}\left(\phi\right)\text{ for each }z\in\gamma\}.$$

One can easily observe that $\bar{S}_\gamma\left(\phi,\Phi\right)$ is closed, convex and

$$(14)\qquad\qquad\qquad \emptyset\neq\bar{S}_\gamma\left(\phi,\Phi\right)\subseteq\bar{U}_\gamma\left(\phi,\Phi\right).$$

Theorem 4 *Assume that the path γ is piecewise smooth and $f:\gamma\to\mathbb{C}$ is continuous on γ and there exist ϕ, $\Phi\in\mathbb{C}$ with $\phi\neq\Phi$ such that $f\in\bar{\Delta}_\gamma\left(\phi,\Phi\right)$. Then*
$$(15)$$
$$\mathcal{P}_\gamma\left(f,\overline{f}\right)\leq\operatorname{Re}\left[\left(\Phi-\frac{1}{\ell\left(\gamma\right)}\int_\gamma f\left(z\right)\left|dz\right|\right)\left(\frac{1}{\ell\left(\gamma\right)}\int_\gamma\overline{f\left(z\right)}\left|dz\right|-\overline{\phi}\right)\right]\leq\frac{1}{4}\left|\Phi-\phi\right|^2$$

or, equivalently,

$$(16)\qquad \mathcal{P}_\gamma\left(f,\overline{f}\right)\leq\frac{1}{4}\left|\Phi-\phi\right|^2-\left|\frac{1}{\ell\left(\gamma\right)}\int_\gamma f\left(z\right)\left|dz\right|-\frac{\phi+\Phi}{2}\right|^2\leq\frac{1}{4}\left|\Phi-\phi\right|^2.$$

Proof. Since $f\in\bar{\Delta}_\gamma\left(\phi,\Phi\right)$, hence

$$\operatorname{Re}\left[\left(\Phi-f\left(z\right)\right)\left(\overline{f\left(z\right)}-\overline{\phi}\right)\right]\geq 0\text{ for each }z\in\gamma.$$

Therefore

$$\frac{1}{\ell\left(\gamma\right)}\int_\gamma\operatorname{Re}\left[\left(\Phi-f\left(z\right)\right)\left(\overline{f\left(z\right)}-\overline{\phi}\right)\right]\left|dz\right|\geq 0$$

and by Lemma 1 we deduce the desired result.

We have:

Lemma 5 *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ, then for all $\lambda \in \mathbb{C}$ we have*

$$(17) \qquad \mathcal{P}_\gamma\left(f, \overline{f}\right) = \frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\right)\left(\overline{f(v)} - \lambda\right)|dv|.$$

Proof. We observe that

$$\frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\right)\left(\overline{f(v)} - \lambda\right)|dv|$$

$$= \frac{1}{\ell(\gamma)} \int_\gamma |f(v)|^2\,|dv| - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\,\frac{1}{\ell(\gamma)} \int_\gamma \overline{f(v)}\,|dv|$$

$$- \lambda \frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\right)|dv|$$

$$= \frac{1}{\ell(\gamma)} \int_\gamma |f(v)|^2\,|dv| - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\,\frac{1}{\ell(\gamma)} \overline{\left(\int_\gamma f(v)\,|dv|\right)}$$

$$= \frac{1}{\ell(\gamma)} \int_\gamma |f(v)|^2\,|dv| - \left|\frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\right|^2$$

for any $\lambda \in \mathbb{C}$, which proves (17).

We have:

Lemma 6 *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ and there exists $c \in \mathbb{C}$ and $\rho > 0$ such that*

$$(18) \qquad\qquad f \in \overline{D}(c, \rho) := \{z \in \mathbb{C}|\ |z - c| \le \rho\},$$

then

$$(19) \qquad 0 \le \mathcal{P}_\gamma\left(f, \overline{f}\right) \le \rho \frac{1}{\ell(\gamma)} \left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\right||dv|$$

and

$$(20) \qquad 0 \le \mathcal{P}_\gamma^2\left(f, \overline{f}\right) \le \rho^2 \left(\frac{1}{\ell(\gamma)} \left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\right||dv|\right)^2$$

$$\le \rho^2 \mathcal{P}_\gamma\left(f, \overline{f}\right).$$

Proof. For the equality (17) for $\lambda = \overline{c}$ we have

$$\mathcal{P}_\gamma\left(f, \overline{f}\right) = \left|\frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\right)\left(\overline{f(v)} - \overline{c}\right)|dv|\right|$$

$$\le \frac{1}{\ell(\gamma)} \left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\right|\left|\overline{f(v)} - \overline{c}\right||dv|$$

$$= \frac{1}{\ell(\gamma)} \left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\right||f(v) - c|\,|dv|$$

$$\le \rho \frac{1}{\ell(\gamma)} \left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz|\right||dv|,$$

which proves (19).

Using Cauchy-Bunyakovsky-Schwarz integral inequality, we have

(21)
$$\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|\,|dv|$$

$$\leq \left(\frac{1}{\ell(\gamma)}\int_\gamma \left|f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2\,|dv|\right)^{1/2}$$

$$= \left(\frac{1}{\ell(\gamma)}\int_\gamma |f(z)|^2\,|dz| - \left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2\right)^{1/2},$$

where for the last equality we used (8).

From (19) and (21) we have

$$0 \leq \mathcal{P}_\gamma\left(f,\overline{f}\right) \leq \rho\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|\,|dv| \leq \rho\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2},$$

which implies (20).

Corollary 7 *Assume that the path γ is piecewise smooth and $f : \gamma \to \mathbb{C}$ is continuous on γ and there exist $\phi, \Phi \in \mathbb{C}$ with $\phi \neq \Phi$ such that $f \in \bar{\Delta}_\gamma(\phi, \Phi)$. Then*

(22)
$$0 \leq \mathcal{P}_\gamma\left(f,\overline{f}\right) \leq \frac{1}{2}|\Phi - \phi|\frac{1}{\ell(\gamma)}\left|\int_\gamma f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|\,|dv|$$

and

(23)
$$0 \leq \mathcal{P}_\gamma^2\left(f,\overline{f}\right) \leq \frac{1}{4}|\Phi - \phi|^2\left(\frac{1}{\ell(\gamma)}\int_\gamma\left|f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|\,|dv|\right)^2$$

$$\leq \frac{1}{4}|\Phi - \phi|^2\,\mathrm{Re}\left[\left(\Phi - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right)\left(\frac{1}{\ell(\gamma)}\int_\gamma \overline{f(z)}\,|dz| - \overline{\phi}\right)\right]$$

$$= \frac{1}{4}|\Phi - \phi|^2\left[\frac{1}{4}|\Phi - \phi|^2 - \left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz| - \frac{\phi + \Phi}{2}\right|^2\right]$$

$$\leq \frac{1}{16}|\Phi - \phi|^4.$$

3   Refinements of Grüss' Inequality

We start with the following identity of interest:

Lemma 8 *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, then*

$$(24) \quad \mathcal{D}_\gamma(f,g) = \frac{1}{2(w-u)^2} \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dw \right) dz$$

$$= \frac{1}{2(w-u)^2} \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dz \right) dw$$

$$= \frac{1}{2(w-u)^2} \int_\gamma \int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dz \, dw.$$

Proof. For any $z \in \gamma$ the integral $\int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dw$ exists and

$$I(z) := \int_\gamma (f(z) - f(w))(g(z) - g(w)) \, dw$$

$$= \int_\gamma (f(z)g(z) + f(w)g(w) - g(z)f(w) - f(z)g(w)) \, dw$$

$$= f(z)g(z) \int_\gamma dw + \int_\gamma f(w)g(w) \, dw - g(z) \int_\gamma f(w) \, dw - f(z) \int_\gamma g(w) \, dw$$

$$= (w-u)f(z)g(z) + \int_\gamma f(w)g(w) \, dw - g(z) \int_\gamma f(w) \, dw - f(z) \int_\gamma g(w) \, dw.$$

The function $I(z)$ is also continuous on γ, then the integral $\int_\gamma I(z) \, dz$ exists and

$$\int_\gamma I(z) \, dz = \int_\gamma \left[(w-u)f(z)g(z) + \int_\gamma f(w)g(w) \, dw \right.$$

$$\left. -g(z) \int_\gamma f(w) \, dw - f(z) \int_\gamma g(w) \, dw \right] dz$$

$$= (w-u) \int_\gamma f(z)g(z) \, dz + (w-u) \int_\gamma f(w)g(w) \, dw$$

$$- \int_\gamma f(w) \, dw \int_\gamma g(z) \, dz - \int_\gamma g(w) \, dw \int_\gamma f(z) \, dz$$

$$= 2(w-u) \int_\gamma f(z)g(z) \, dz - 2 \int_\gamma f(z) \, dz \int_\gamma g(z) \, dz = 2(w-u)^2 \mathcal{P}_\gamma(f,g),$$

which proves the first equality in (24).

The rest follows in a similar manner and we omit the details.

We have:

Lemma 9 *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, then*

$$(25) \qquad |\mathcal{D}_\gamma(f,g)| \leq \epsilon^2(\gamma) \left[\mathcal{P}_\gamma(f, \overline{f}) \right]^{1/2} \left[\mathcal{P}_\gamma(g, \overline{g}) \right]^{1/2},$$

where $\epsilon(\gamma) := \frac{\ell(\gamma)}{|w-u|}$ can be interpreted as the deviation of the path γ from the segment joining the points u and w in $\mathbb{C}$.

Proof. Taking the modulus in the first equality in (24), we get

$$|\mathcal{D}_\gamma\left(f,g\right)| = \frac{1}{2\left|w-u\right|^2}\left|\int_\gamma\left(\int_\gamma\left(f\left(z\right)-f\left(w\right)\right)\left(g\left(z\right)-g\left(w\right)\right)dw\right)dz\right|$$

$$\leq \frac{1}{2\left|w-u\right|^2}\int_\gamma\left|\int_\gamma\left(f\left(z\right)-f\left(w\right)\right)\left(g\left(z\right)-g\left(w\right)\right)dw\right||dz| =: A$$

Using the Cauchy-Bunyakovsky-Schwarz integral inequality, we have

$$\left|\int_\gamma\left(f\left(z\right)-f\left(w\right)\right)\left(g\left(z\right)-g\left(w\right)\right)dw\right|$$

$$\leq \left(\int_\gamma|f\left(z\right)-f\left(w\right)|^2|dw|\right)^{1/2}\left(\int_\gamma|g\left(z\right)-g\left(w\right)|^2|dw|\right)^{1/2},$$

which implies that

$$A$$

$$\leq \frac{1}{2\left|w-u\right|^2}\int_\gamma\left(\int_\gamma|f\left(z\right)-f\left(w\right)|^2|dw|\right)^{1/2}\left(\int_\gamma|g\left(z\right)-g\left(w\right)|^2|dw|\right)^{1/2}|dz|$$

$$=: B.$$

By the Cauchy-Bunyakovsky-Schwarz integral inequality, we also have

$$\int_\gamma\left(\int_\gamma|f\left(z\right)-f\left(w\right)|^2|dw|\right)^{1/2}\left(\int_\gamma|g\left(z\right)-g\left(w\right)|^2|dw|\right)^{1/2}|dz|$$

$$\leq \left(\int_\gamma\left[\left(\int_\gamma|f\left(z\right)-f\left(w\right)|^2|dw|\right)^{1/2}\right]^2|dz|\right)^{1/2}$$

$$\times \left(\int_\gamma\left[\left(\int_\gamma|g\left(z\right)-g\left(w\right)|^2|dw|\right)^{1/2}\right]^2|dz|\right)^{1/2}$$

$$= \left(\int_\gamma\left(\int_\gamma|f\left(z\right)-f\left(w\right)|^2|dw|\right)|dz|\right)^{1/2}\left(\int_\gamma\left(\int_\gamma|g\left(z\right)-g\left(w\right)|^2|dw|\right)|dz|\right)^{1/2},$$

which implies that

$$(26)\quad B \leq \frac{1}{2\left|w-u\right|^2}\left(\int_\gamma\left(\int_\gamma|f\left(z\right)-f\left(w\right)|^2|dw|\right)|dz|\right)^{1/2}$$

$$\times \left(\int_\gamma\left(\int_\gamma|g\left(z\right)-g\left(w\right)|^2|dw|\right)|dz|\right)^{1/2}.$$

Now, observe that

$$(27) \quad \int_\gamma \left(\int_\gamma |f(z) - f(w)|^2 \, |dw| \right) |dz|$$

$$= \int_\gamma \left(\int_\gamma \left(|f(z)|^2 - 2\operatorname{Re}\left(f(z) \overline{f(w)} \right) + |f(w)|^2 \right) |dw| \right) |dz|$$

$$= \int_\gamma \left(\ell(\gamma) |f(z)|^2 - 2\operatorname{Re}\left(f(z) \int_\gamma \overline{f(w)} \, |dw| \right) + \int_\gamma |f(w)|^2 \, |dw| \right) |dz|$$

$$= \ell(\gamma) \int_\gamma |f(z)|^2 \, |dz| - 2\operatorname{Re}\left(\int_\gamma f(z) \, |dz| \int_\gamma \overline{f(w)} \, |dw| \right) + \ell(\gamma) \int_\gamma |f(w)|^2 \, |dw|$$

$$= 2\ell(\gamma) \int_\gamma |f(z)|^2 \, |dz| - 2\operatorname{Re}\left(\int_\gamma f(z) \, |dz| \, \overline{\left(\int_\gamma f(w) \, |dw| \right)} \right)$$

$$= 2 \left[\ell(\gamma) \int_\gamma |f(z)|^2 \, |dz| - \left| \int_\gamma f(z) \, |dz| \right|^2 \right] = 2\ell^2(\gamma) \, \mathcal{P}_\gamma \left(f, \overline{f} \right)$$

and, similarly

$$(28) \qquad \int_\gamma \left(\int_\gamma |g(z) - g(w)|^2 \, |dw| \right) |dz| = 2\ell^2(\gamma) \, \mathcal{P}_\gamma \left(g, \overline{g} \right).$$

Making use of (27) and (28), we get

$$B \leq \frac{1}{2\,|w - u|^2} \left[2\ell^2(\gamma) \, \mathcal{P}_\gamma \left(f, \overline{f} \right) \right]^{1/2} \left[2\ell^2(\gamma) \, \mathcal{P}_\gamma \left(g, \overline{g} \right) \right]^{1/2}$$

$$= \frac{\ell^2(\gamma)}{|w - u|^2} \left[\mathcal{P}_\gamma \left(f, \overline{f} \right) \right]^{1/2} \left[\mathcal{P}_\gamma \left(g, \overline{g} \right) \right]^{1/2},$$

which proves the desired result (25).

Remark 1 *For $g = f$ we have*

$$(29) \qquad \mathcal{D}_\gamma(f, f) = \frac{1}{w - u} \int_\gamma f^2(z) \, dz - \left(\frac{1}{w - u} \int_\gamma f(z) \, dz \right)^2$$

and by (25) we get

$$(30) \qquad |\mathcal{D}_\gamma(f, f)| \leq \epsilon^2(\gamma) \, \mathcal{P}_\gamma \left(f, \overline{f} \right).$$

For $g = \overline{f}$ we have

$$(31) \qquad \mathcal{D}_\gamma\left(f, \overline{f} \right) = \frac{1}{w - u} \int_\gamma |f(z)|^2 \, dz - \frac{1}{w - u} \int_\gamma f(z) \, dz \frac{1}{w - u} \int_\gamma \overline{f(z)} \, dz$$

and by (25) we get

$$(32) \qquad \left| \mathcal{D}_\gamma\left(f, \overline{f} \right) \right| \leq \epsilon^2(\gamma) \, \mathcal{P}_\gamma \left(f, \overline{f} \right).$$

Theorem 10 *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ and there exists ϕ, Φ, ψ, $\Psi \in \mathbb{C}$, $\phi \neq \Phi$, $\psi \neq \Psi$ such that $f \in \bar{\Delta}_\gamma(\phi, \Phi)$ and $g \in \bar{\Delta}_\gamma(\psi, \Psi)$ then*

$$
(33) \quad |\mathcal{D}_\gamma(f, g)| \leq \epsilon^2(\gamma) \left(\frac{1}{4} |\Phi - \phi|^2 - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right|^2 \right)^{1/2}
$$

$$
\times \left(\frac{1}{4} |\Psi - \psi|^2 - \left| \frac{1}{\ell(\gamma)} \int_\gamma g(z) |dz| - \frac{\psi + \Psi}{2} \right|^2 \right)^{1/2}
$$

$$
\leq \epsilon^2(\gamma) \left[\frac{1}{4} |\Phi - \phi| |\Psi - \psi| \right.
$$

$$
\left. - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right| \left| \frac{1}{\ell(\gamma)} \int_\gamma g(z) |dz| - \frac{\psi + \Psi}{2} \right| \right]
$$

$$
\leq \frac{1}{4} \epsilon^2(\gamma) |\Phi - \phi| |\Psi - \psi|.
$$

Proof. The first inequality in (33) follows by Corollary 7 and Lemma 9.

Using the elementary inequality

$$
\left(m^2 - n^2 \right)^{1/2} \left(p^2 - q^2 \right)^{1/2} \leq mp - nq
$$

that holds for $m \geq n \geq 0$ and $p \geq q \geq 0$, for the choices

$$
m = \frac{1}{2} |\Phi - \phi|, \ n = \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right|
$$

and

$$
p = \frac{1}{2} |\Psi - \psi|, \ q = \left| \frac{1}{\ell(\gamma)} \int_\gamma g(z) |dz| - \frac{\psi + \Psi}{2} \right|
$$

we get the second inequality in (33).

The last part is obvious.

Remark 2 *If there is information on the boundedness of only one function, namely $f \in \bar{\Delta}_\gamma(\phi, \Phi)$, where $g : \gamma \to \mathbb{C}$ is continuous, then we have the "premature" Grüss inequality*

$$
(34) \quad |\mathcal{D}_\gamma(f, g)| \leq \epsilon^2(\gamma) \left(\frac{1}{4} |\Phi - \phi|^2 - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z) |dz| - \frac{\phi + \Phi}{2} \right|^2 \right)^{1/2}
$$

$$
\times \left[\mathcal{P}_\gamma(g, \overline{g}) \right]^{1/2}.
$$

4   Applications for Trapezoid Inequality

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ and f holomorphic in G, an open domain and suppose $\gamma \subset G$. Put $z(a) = u$ and $z(b) = w$ with u, $w \in \mathbb{C}$. Using the integration by parts formula (6) twice we have

$$\int_{\gamma_{u,v}} (z - v) f'(z)\, dz = (v - u) f(u) - \int_{\gamma_{u,v}} f(z)\, dz$$

and

$$\int_{\gamma_{v,w}} (z - v) f'(z)\, dz = (w - v) f(w) - \int_{\gamma_{v,w}} f(z)\, dz,$$

for any $v \in \gamma$.

If we add these two equalities, we get the *generalized trapezoid equality*

$$(35) \quad (v - u) f(u) + (w - v) f(w) - \int_{\gamma} f(z)\, dz$$

$$= \int_{\gamma_{u,v}} (z - v) f'(z)\, dz + \int_{\gamma_{v,w}} (z - v) f'(z)\, dz = \int_{\gamma} (z - v) f'(z)\, dz$$

with the above assumptions for u, v and w on γ.

If we take $v = \frac{u+w}{2}$, then we get the *trapezoid equality*

$$(36) \qquad (w - u) \frac{f(u) + f(w)}{2} - \int_{\gamma} f(z)\, dz = \int_{\gamma} \left(z - \frac{u+w}{2} \right) f'(z)\, dz,$$

which also can be written as

$$\frac{f(u) + f(w)}{2} - \frac{1}{w - u} \int_{\gamma} f(z)\, dz = \mathcal{D}_{\gamma} \left(f', h \right)$$

where $h(z) = z - \frac{u+w}{2}$, $z \in \mathbb{C}$, since $\int_{\gamma} \left(z - \frac{u+w}{2} \right) dz = 0$.

If $f' \in \bar{\Delta}_{\gamma}(\theta, \Theta)$ for some θ, $\Theta \in \mathbb{C}$ with $\theta \neq \Theta$, then by (34) we get the following trapezoid inequality of interest:

$$\left| \frac{f(u) + f(w)}{2} - \frac{1}{w - u} \int_{\gamma} f(z)\, dz \right| = \mathcal{D}_{\gamma} \left(f', h \right)$$

$$\leq \epsilon^2(\gamma) \left(\frac{1}{4} |\Theta - \theta|^2 - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} f'(z)\, |dz| - \frac{\theta + \Theta}{2} \right|^2 \right)^{1/2} \left[\mathcal{P}_{\gamma} \left(h, \overline{h} \right) \right]^{1/2}$$

$$\leq \frac{1}{2} |\Theta - \theta| \, \epsilon^2(\gamma) \left[\mathcal{P}_{\gamma} \left(h, \overline{h} \right) \right]^{1/2},$$

where

$$\mathcal{P}_{\gamma} \left(h, \overline{h} \right) = \frac{1}{\ell(\gamma)} \int_{\gamma} \left| z - \frac{u+w}{2} \right|^2 |dz| - \left| \frac{1}{\ell(\gamma)} \int_{\gamma} \left(z - \frac{u+w}{2} \right) |dz| \right|^2.$$

If the path γ is a segment $[u,w] \subset G$ connecting two distinct points u and w in G then we write $\int_\gamma f(z)\, dz$ as $\int_u^w f(z)\, dz$. We then have

$$\epsilon(\gamma) = \frac{\ell(\gamma)}{|w-u|} = \frac{|w-u|}{|w-u|} = 1,$$

$$\frac{1}{\ell([u,w])} \int_u^w f'(z)\,|dz| = \frac{|w-u|}{|w-u|} \int_0^1 f'((1-t)u+tw)\, dt = \frac{f(w)-f(u)}{w-u}$$

and

$$\mathcal{P}_{[u,w]}\left(h,\overline{h}\right) = \frac{|w-u|}{|w-u|} \int_0^1 \left|(1-t)u+tw - \frac{u+w}{2}\right|^2 dt$$

$$- \left|\frac{|w-u|}{|w-u|} \int_0^1 \left((1-t)u+tw - \frac{u+w}{2}\right) dt\right|^2$$

$$= |w-u|^2 \int_0^1 \left(t-\frac{1}{2}\right)^2 dt - |w-u|^2 \left|\int_0^1 \left(t-\frac{1}{2}\right) dt\right|^2 = \frac{1}{12}|w-u|^2.$$

Based on the calculations outlined above, we can state the following result as well:

Proposition 11 *Assume that f is holomorphic in G, an open domain and suppose $[u,w] \subset G$. If $f' \in \overline{\Delta}_{[u,w]}(\theta,\Theta)$ for some θ, $\Theta \in \mathbb{C}$ with $\theta \neq \Theta$, then*

$$(37) \quad \left|\frac{f(u)+f(w)}{2} - \frac{1}{w-u}\int_u^w f(z)\, dz\right|$$

$$\leq \frac{\sqrt{3}}{6}|w-u|\left(\frac{1}{4}|\Theta-\theta|^2 - \left|\frac{f(w)-f(u)}{w-u} - \frac{\theta+\Theta}{2}\right|^2\right)^{1/2}$$

$$\leq \frac{\sqrt{3}}{12}|w-u|\,|\Theta-\theta|.$$

Since

$$\mathrm{Re}\left[\left(\Phi - \frac{f(w)-f(u)}{w-u}\right)\left(\overline{\frac{f(w)-f(u)}{w-u}} - \overline{\phi}\right)\right]$$

$$= \frac{1}{4}|\Phi-\phi|^2 - \left|\frac{f(w)-f(u)}{w-u} - \frac{\phi+\Phi}{2}\right|^2,$$

then the inequality (37) can also be written as

$$(38) \quad \left|\frac{f(u)+f(w)}{2} - \frac{1}{w-u}\int_u^w f(z)\, dz\right|$$

$$\leq \frac{\sqrt{3}}{6}|w-u|\left(\mathrm{Re}\left[\left(\Phi - \frac{f(w)-f(u)}{w-u}\right)\left(\overline{\frac{f(w)-f(u)}{w-u}} - \overline{\phi}\right)\right]\right)^{1/2}$$

$$\leq \frac{\sqrt{3}}{12}|w-u|\,|\Theta-\theta|.$$

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