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Functions With Applications*

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New Inequalities of Fejér and Hermite-Hadamard type Concerning Convex and Quasi-Convex Functions With Applications

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Abstract.: This research contains new integral inequalities of Fejér and Hermite-Hadamard type involving convex and quasi-convex functions. Applications of the newly established results for special means of positive real numbers are given.

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1. INTRODUCTION

We should mention that \mathcal{U} is an interval and \mathcal{U}° is the interior of \mathcal{U} wherever they appear in this paper.

The following definitions are well known in the literature.

Definition 1.1. [21] A function $\gamma_1 : \mathcal{U} \subset \mathcal{R} \rightarrow \mathcal{R}$ is called convex function (in the classical sense) if the inequality

$$\gamma_1(\alpha\lambda + (1-\alpha)\mu) \leq \alpha\gamma_1(\lambda) + (1-\alpha)\gamma_1(\mu)$$

holds for all $\lambda, \mu \in \mathcal{U}$ and $\alpha \in [0, 1]$.

Definition 1.2. [21] A function $\gamma_1 : \mathcal{U} \subset \mathcal{R} \rightarrow \mathcal{R}$ is called quasi convex function, if the inequality

$$\gamma_1(\alpha\lambda + (1 - \alpha)\mu) \leq \max\{\gamma_1(\lambda), \gamma_1(\mu)\}$$

holds for all $\lambda, \mu \in \mathcal{U}$ and $\alpha \in [0, 1]$.

It should be noted that a convex function must be a quasi-convex function but not conversely. The past few decades have witnessed remarkable research on inequalities, including a large number of papers and many fertile applications. The subject has evoked considerable interest from many mathematicians, and an extensive number of new results have been studied in the literature. It is recognized that in general some specific inequalities provide a useful and essential contrivance in the growth of various branches of mathematics. A number of interesting results have been proved by using the concept of classical convexity, s -convexity and harmonically s -convex functions, see for instance [1]-[28] and the references therein. Here we recall some of the results for convex and quasi-convex functions which are closely related to the research of our paper.

Dragomir and Agarwal [4] proved the subsequent result for differentiable convex mappings.

Theorem 1.3. [4] Let $\gamma_1 : \mathcal{U}^\circ \subseteq \mathcal{R} \rightarrow \mathcal{R}$ be a differentiable mapping on \mathcal{U}° and $\theta_1, \theta_2 \in \mathcal{U}^\circ$ with $\theta_1 < \theta_2$. If $|\gamma_1'|$ is convex on $[\theta_1, \theta_2]$, then

$$\left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} - \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \gamma_1(\lambda) d\lambda \right| \leq \frac{(\theta_2 - \theta_1)}{8} [|\gamma_1'(\theta_1)| + |\gamma_1'(\theta_2)|] \quad (1.1)$$

Hwang [6] obtained the given results which contains the result of Theorem 1.3 as a special case.

Theorem 1.4. [6] Let $\gamma_1 : \mathcal{U} \subseteq \mathcal{R} \rightarrow \mathcal{R}$ be a differentiable mapping on \mathcal{U}° and $\gamma_2 : [\theta_1, \theta_2] \rightarrow [0, \infty)$ be a continuous and symmetric mapping with respect to $\frac{\theta_1 + \theta_2}{2}$, where $\theta_1, \theta_2 \in \mathcal{U}^\circ$ with $\theta_1 < \theta_2$.

(1) If $\gamma_1' \in \mathcal{L}_1([\theta_1, \theta_2])$ and $|\gamma_1'|$ is convex on $[\theta_1, \theta_2]$, then

$$\left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \leq \frac{(\theta_2 - \theta_1)}{2} \left[\frac{|\gamma_1'(\theta_1)| + |\gamma_1'(\theta_2)|}{2} \right] \int_0^1 \int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda) d\lambda d\alpha. \quad (1.2)$$

(2) If $\gamma_1' \in \mathcal{L}_1([\theta_1, \theta_2])$ and $|\gamma_1'|^q$ is convex on $[\theta_1, \theta_2]$ for $q \geq 1$, then

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \frac{(\theta_2 - \theta_1)}{2} \left[\frac{|\gamma_1'(\theta_1)|^q + |\gamma_1'(\theta_2)|^q}{2} \right]^{\frac{1}{q}} \int_0^1 \int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda) d\lambda d\alpha, \quad (1.3) \end{aligned}$$

(3) If $\gamma_1' \in \mathcal{L}_1([\theta_1, \theta_2])$ and $|\gamma_1'|$ is quasi-convex on $[\theta_1, \theta_2]$, then

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \frac{(\theta_2 - \theta_1)}{4} \left[\max \left\{ |\gamma_1'(\theta_1)|, \left| \gamma_1' \left(\frac{\theta_1 + \theta_2}{2} \right) \right| \right\} \right. \\ & \quad \left. + \max \left\{ \left| \gamma_1' \left(\frac{\theta_1 + \theta_2}{2} \right) \right|, |\gamma_1'(\theta_2)| \right\} \right] \int_0^1 \int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda) d\lambda d\alpha. \quad (1.4) \end{aligned}$$

(4) If $\gamma_1' \in \mathcal{L}_1([\theta_1, \theta_2])$ and $|\gamma_1'|^q$ is quasi-convex on $[\theta_1, \theta_2]$ for $q \geq 1$, then

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \frac{(\theta_2 - \theta_1)}{4} \left[\left(\max \left\{ |\gamma_1'(\theta_1)|^q, \left| \gamma_1' \left(\frac{\theta_1 + \theta_2}{2} \right) \right|^q \right\} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\max \left\{ \left| \gamma_1' \left(\frac{\theta_1 + \theta_2}{2} \right) \right|^q, |\gamma_1'(\theta_2)|^q \right\} \right)^{\frac{1}{q}} \right] \int_0^1 \int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda) d\lambda d\alpha. \quad (1.5) \end{aligned}$$

where

$$T(\alpha) = \frac{1+\alpha}{2}\theta_1 + \frac{1-\alpha}{2}\theta_2 \text{ and } S(\alpha) = \frac{1-\alpha}{2}\theta_1 + \frac{1+\alpha}{2}\theta_2.$$

The subsequent results are due to Hua et al. [7].

Theorem 1.5. [7] Let $\gamma_1 : \mathcal{U} \subseteq \mathcal{R} \rightarrow \mathcal{R}$ be a differentiable mapping on \mathcal{U}° and $\gamma_2 : [\theta_1, \theta_2] \rightarrow [0, \infty)$ be a continuous and symmetric mapping with respect to $\frac{\theta_1 + \theta_2}{2}$, where $\theta_1, \theta_2 \in \mathcal{U}^\circ$ with $\theta_1 < \theta_2$.

(1) If $\gamma'_1 \in \mathcal{L}_1([\theta_1, \theta_2])$ and $|\gamma'_1|^\mathbf{q}$ is convex on $[\theta_1, \theta_2]$ for $\mathbf{q} \geq 1$, then

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \frac{(\theta_2 - \theta_1)}{2} \left[\frac{|\gamma'_1(\theta_1)|^\mathbf{q} + |\gamma'_1(\theta_2)|^\mathbf{q}}{2} \right]^{\frac{1}{\mathbf{q}}} \int_0^1 \int_{M(\alpha)}^{V(\alpha)} \gamma_2(\lambda) d\lambda d\alpha, \quad (1.6) \end{aligned}$$

(2) If $\gamma'_1 \in \mathcal{L}_1([\theta_1, \theta_2])$ and $|\gamma'_1|^\mathbf{q}$ is convex on $[\theta_1, \theta_2]$ for $\mathbf{q} > 1$, then

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \frac{(\theta_2 - \theta_1)}{4} \left\{ \left(\frac{3|\gamma'_1(\theta_1)|^\mathbf{q} + |\gamma'_1(\theta_2)|^\mathbf{q}}{4} \right)^{\frac{1}{\mathbf{q}}} + \left(\frac{|\gamma'_1(\theta_1)|^\mathbf{q} + 3|\gamma'_1(\theta_2)|^\mathbf{q}}{4} \right)^{\frac{1}{\mathbf{q}}} \right\} \\ & \times \min \left\{ \left[\int_0^1 \left(\int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda) d\lambda \right)^{\frac{\mathbf{q}}{\mathbf{q}-1}} d\alpha \right]^{1-\frac{1}{\mathbf{q}}}, \left[\int_0^1 \left(\int_{M(\alpha)}^{V(\alpha)} \gamma_2(\lambda) d\lambda \right)^{\frac{\mathbf{q}}{\mathbf{q}-1}} d\alpha \right]^{1-\frac{1}{\mathbf{q}}} \right\}, \quad (1.7) \end{aligned}$$

where

$$M(\alpha) = \alpha\theta_1 + (1-\alpha)\frac{\theta_1 + \theta_2}{2}, \quad V(\alpha) = \alpha\theta_2 + (1-\alpha)\frac{\theta_1 + \theta_2}{2}$$

and $T(\alpha)$ and $S(\alpha)$ are defined as in Theorem 1.4.

Motivated by the results mentioned above the main objective of this paper is prove new results of Fejér and Hermite-Hadamard type by using the convexity and quasi-convexity based new family identities for a positive integer \mathbf{m} . The results proved in this paper may have some relation with those proved earlier for some specific values of \mathbf{m} .

2. NEW RESULTS

An important lemma to prove the results is given as follows.

Lemma 2.1. Let $\gamma_1 : \mathcal{U} \subseteq \mathcal{R} \rightarrow \mathcal{R}$ be a differentiable mapping on \mathcal{U}° and $\gamma_2 : [\theta_1, \theta_2] \rightarrow [0, \infty)$ be a continuous and symmetric mapping with respect to $\frac{\theta_1 + \theta_2}{2}$, where $\theta_1, \theta_2 \in \mathcal{U}^\circ$ with $\theta_1 < \theta_2$. If $\gamma'_1 \in \mathcal{L}_1([\theta_1, \theta_2])$, then the equality

$$\begin{aligned} & \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \\ & = \left(\frac{\theta_2 - \theta_1}{2\mathbf{m}} \right) \left[\int_0^{\frac{\mathbf{m}}{2}} \left(\int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda \right) [\gamma'_1(\zeta_2(\alpha, \mathbf{m})) - \gamma'_1(\zeta_1(\alpha, \mathbf{m}))] d\alpha \right] \quad (2.8) \end{aligned}$$

holds, where $\zeta_1(\alpha, \mathbf{m}) = \frac{\alpha}{\mathbf{m}}\theta_2 + \left(\frac{\mathbf{m}-\alpha}{\mathbf{m}}\right)\theta_1$, $\zeta_2(\alpha, \mathbf{m}) = \frac{\alpha}{\mathbf{m}}\theta_1 + \left(\frac{\mathbf{m}-\alpha}{\mathbf{m}}\right)\theta_2$, $\|\gamma_2\|_\infty = \sup_{\alpha \in [\theta_1, \theta_2]} |\gamma_2(\alpha)|$ and \mathbf{m} is a positive integer.

Proof. Using integration by parts, we have

$$\begin{aligned} \mathcal{U}_1 &= \int_0^{\frac{\mathbf{m}}{2}} \left(\int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda \right) \gamma_1'(\zeta_1(\alpha, \mathbf{m})) d\alpha \\ &= \frac{\mathbf{m}}{\theta_2 - \theta_1} \int_0^1 \left(\int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda \right) d(\gamma_1(\zeta_1(\alpha, \mathbf{m}))) \\ &= \frac{\mathbf{m}}{\theta_2 - \theta_1} \left(\int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda \right) \gamma_1(\zeta_1(\alpha, \mathbf{m})) \Big|_0^{\frac{\mathbf{m}}{2}} \\ &\quad + \int_0^{\frac{\mathbf{m}}{2}} [\gamma_2(\zeta_2(\alpha, \mathbf{m})) + \gamma_2(\zeta_1(\alpha, \mathbf{m}))] \gamma_1(\zeta_1(\alpha, \mathbf{m})) d\alpha \end{aligned}$$

Since γ_2 is symmetric with respect to $\frac{\theta_1 + \theta_2}{2}$, we have

$$\gamma_2(\zeta_2(\alpha, \mathbf{m})) = \gamma_2(\zeta_1(\alpha, \mathbf{m})),$$

and hence

$$\mathcal{U}_1 = -\frac{\mathbf{m}}{\theta_2 - \theta_1} \left(\int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda \right) \gamma_1(\theta_1) + 2 \int_0^{\frac{\mathbf{m}}{2}} \gamma_2(\zeta_1(\alpha, \mathbf{m})) \gamma_1(\zeta_1(\alpha, \mathbf{m})) d\alpha.$$

Setting $\zeta_1(\alpha, \mathbf{m}) = \lambda$, we have

$$\mathcal{U}_1 = -\frac{\mathbf{m}}{\theta_2 - \theta_1} \left(\int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda \right) \gamma_1(\theta_1) + \frac{2\mathbf{m}}{\theta_2 - \theta_1} \int_{\theta_1}^{\frac{\theta_1 + \theta_2}{2}} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda. \quad (2.9)$$

Similarly,

$$\mathcal{U}_2 = \int_0^{\frac{\mathbf{m}}{2}} \left(\int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda \right) \gamma_1'(\zeta_2(\alpha, \mathbf{m})) d\alpha \quad (2.10)$$

$$= \frac{\mathbf{m}}{\theta_2 - \theta_1} \left(\int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda \right) \gamma_1(\theta_2) - \frac{2\mathbf{m}}{\theta_2 - \theta_1} \int_{\frac{\theta_1 + \theta_2}{2}}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda. \quad (2.11)$$

Subtracting (2.9) from (2.10) and multiplying the resulting equality by $\frac{\theta_2 - \theta_1}{2\mathbf{m}}$, we get (2.8). \square

Remark 2.2. If $\mathbf{m} = 2$ in Lemma 2.1, the following identity

$$\begin{aligned} & \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \\ &= \left(\frac{\theta_2 - \theta_1}{4} \right) \left[\int_0^1 \left(\int_{\zeta_1(\alpha, 2)}^{\zeta_2(\alpha, 2)} \gamma_2(\lambda) d\lambda \right) [\gamma_1'(\zeta_2(\alpha, 2)) - \gamma_1'(\zeta_1(\alpha, 2))] d\alpha \right] \end{aligned} \quad (2.12)$$

holds, where $\zeta_1(\alpha, 2) = \frac{\alpha}{2}\theta_2 + (\frac{2-\alpha}{2})\theta_1$, $\zeta_2(\alpha, 2) = \frac{\alpha}{2}\theta_1 + (\frac{2-\alpha}{2})\theta_2$, $\|\gamma_2\|_\infty = \sup_{\alpha \in [\theta_1, \theta_2]} |\gamma_2(\alpha)|$.

Remark 2.3. If $\gamma_2(\lambda) = \frac{1}{\theta_2 - \theta_1}$ for all $\lambda \in [\theta_1, \theta_2]$ in Lemma 2.1, we get the equality

$$\begin{aligned} & \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} - \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \gamma_1(\lambda) d\lambda \\ &= \left(\frac{\theta_2 - \theta_1}{2\mathbf{m}^2} \right) \int_0^{\frac{\mathbf{m}}{2}} (\mathbf{m} - 2\alpha) [\gamma_1'(\zeta_2(\alpha, \mathbf{m})) - \gamma_1'(\zeta_1(\alpha, \mathbf{m}))] d\alpha, \end{aligned} \quad (2.13)$$

where $\zeta_1(\alpha, \mathbf{m})$, $\zeta_2(\alpha, \mathbf{m})$ are defined as in Lemma 2.1 and $\mathbf{m} \geq 1$ is positive integer.

Remark 2.4. If $\gamma_2(\lambda) = \frac{1}{\theta_2 - \theta_1}$ for all $\lambda \in [\theta_1, \theta_2]$ in Lemma 2.1 and $\mathbf{m} = 2$, we get the equality

$$\begin{aligned} & \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} - \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \gamma_1(\lambda) d\lambda \\ &= \left(\frac{\theta_2 - \theta_1}{4} \right) \int_0^1 (1 - \alpha) [\gamma_1'(\zeta_2(\alpha, 2)) - \gamma_1'(\zeta_1(\alpha, 2))] d\alpha, \end{aligned} \quad (2.14)$$

where $\zeta_1(\alpha, 2)$, $\zeta_2(\alpha, 2)$ are defined as in Remark 2.2.

Theorem 2.5. Let $\gamma_1 : \mathcal{U} \subseteq \mathcal{R} \rightarrow \mathcal{R}$ be a differentiable mapping on \mathcal{U}° and $\gamma_2 : [\theta_1, \theta_2] \rightarrow [0, \infty)$ be a continuous and symmetric mapping with respect to $\frac{\theta_1 + \theta_2}{2}$, where $\theta_1, \theta_2 \in \mathcal{U}^\circ$ with $\theta_1 < \theta_2$. If $\gamma_1' \in \mathcal{L}_1([\theta_1, \theta_2])$ and $|\gamma_1'|^{\mathbf{q}}$ is convex on $[\theta_1, \theta_2]$ for $\mathbf{q} \geq 1$, then

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \left(\frac{\theta_2 - \theta_1}{\mathbf{m}^{1 - \frac{1}{\mathbf{q}}}} \right) \left[\frac{|\gamma_1'(\theta_1)|^{\mathbf{q}} + |\gamma_1'(\theta_2)|^{\mathbf{q}}}{4} \right]^{\frac{1}{\mathbf{q}}} \int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha, \end{aligned} \quad (2.15)$$

where \mathbf{m} is a positive integer.

Proof. From (2.8) and the power-mean integral inequality, we get

$$\begin{aligned}
 & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\
 & \leq \left(\frac{\theta_2 - \theta_1}{2\mathbf{m}} \right) \left(\int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \right)^{1 - \frac{1}{\mathbf{q}}} \\
 & \quad \times \left\{ \left(\int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \int_0^{\frac{\mathbf{m}}{2}} |\gamma_1'(\zeta_2(\alpha, \mathbf{m}))|^{\mathbf{q}} d\alpha \right)^{\frac{1}{\mathbf{q}}} \right. \\
 & \quad \left. + \left(\int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \int_0^{\frac{\mathbf{m}}{2}} |\gamma_1'(\zeta_1(\alpha, \mathbf{m}))|^{\mathbf{q}} d\alpha \right)^{\frac{1}{\mathbf{q}}} \right\} \\
 & \quad \cdot \quad (2.16)
 \end{aligned}$$

Using the discrete power-mean inequality $\alpha^r + \beta^r \leq 2^{1-r}(\alpha + \beta)^r$ for $\alpha > 0, \beta > 0, 0 < r < 1$ and the convexity of $|\gamma_1'|^{\mathbf{q}}$ on $[\theta_1, \theta_2]$ for $\mathbf{q} \geq 1$, we get

$$\begin{aligned}
 & \left(\int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \int_0^{\frac{\mathbf{m}}{2}} |\gamma_1'(\zeta_2(\alpha, \mathbf{m}))|^{\mathbf{q}} d\alpha \right)^{\frac{1}{\mathbf{q}}} \\
 & \quad + \left(\int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \int_0^{\frac{\mathbf{m}}{2}} |\gamma_1'(\zeta_1(\alpha, \mathbf{m}))|^{\mathbf{q}} d\alpha \right)^{\frac{1}{\mathbf{q}}} \\
 & \leq 2^{1 - \frac{1}{\mathbf{q}}} \left(\int_0^{\frac{\mathbf{m}}{2}} [|\gamma_1'(\zeta_2(\alpha, \mathbf{m}))|^{\mathbf{q}} + |\gamma_1'(\zeta_1(\alpha, \mathbf{m}))|^{\mathbf{q}}] d\alpha \right)^{\frac{1}{\mathbf{q}}} \left(\int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \right)^{\frac{1}{\mathbf{q}}} \\
 & = 2^{1 - \frac{1}{\mathbf{q}}} \left(\frac{\mathbf{m}}{2} \right)^{\frac{1}{\mathbf{q}}} (|\gamma_1'(\theta_1)|^{\mathbf{q}} + |\gamma_1'(\theta_2)|^{\mathbf{q}})^{\frac{1}{\mathbf{q}}} \left(\int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \right)^{\frac{1}{\mathbf{q}}} \quad (2.17)
 \end{aligned}$$

Combining (2.17) and (2.16), we obtain (2.15). \square

Corollary 2.6. *Under the conditions of Theorem 2.5, if $\mathbf{m} = 2$ then*

$$\left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \leq \left(\frac{\theta_2 - \theta_1}{2} \right) \left[\frac{|\gamma_1'(\theta_1)|^q + |\gamma_1'(\theta_2)|^q}{2} \right]^{\frac{1}{q}} \int_0^1 \int_{\zeta_1(\alpha, 2)}^{\zeta_2(\alpha, 2)} \gamma_2(\lambda) d\lambda d\alpha, \quad (2.18)$$

where $\zeta_1(\alpha, 2) = \frac{\alpha}{2}\theta_2 + (\frac{2-\alpha}{2})\theta_1$, $\zeta_2(\alpha, 2) = \frac{\alpha}{2}\theta_1 + (\frac{2-\alpha}{2})\theta_2$.

Corollary 2.7. *Suppose that the assumptions of Theorem 2.5 are fulfilled. If $\mathbf{q} = 1$, then the inequality*

$$\left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \leq \left(\frac{\theta_2 - \theta_1}{2} \right) \left[\frac{|\gamma_1'(\theta_1)| + |\gamma_1'(\theta_2)|}{2} \right] \int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha, \quad (2.19)$$

holds, where \mathbf{m} is a positive integer and $\zeta_1(\alpha, \mathbf{m})$ and $\zeta_2(\alpha, \mathbf{m})$ are defined as in Lemma 2.1.

Corollary 2.8. *If the assumptions of Theorem 2.5 are satisfied and if $\mathbf{q} = 1$, $\mathbf{m} = 2$, the inequality*

$$\left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \leq \left(\frac{\theta_2 - \theta_1}{2} \right) \left[\frac{|\gamma_1'(\theta_1)| + |\gamma_1'(\theta_2)|}{2} \right] \int_0^1 \int_{\zeta_1(\alpha, 2)}^{\zeta_2(\alpha, 2)} \gamma_2(\lambda) d\lambda d\alpha, \quad (2.20)$$

holds, where $\zeta_1(\alpha, 2) = \frac{\alpha}{2}\theta_2 + (\frac{2-\alpha}{2})\theta_1$, $\zeta_2(\alpha, 2) = \frac{\alpha}{2}\theta_1 + (\frac{2-\alpha}{2})\theta_2$.

Corollary 2.9. *If we combine (1. 3), (1. 6) and (2. 18), we get*

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \left(\frac{\theta_2 - \theta_1}{2} \right) \left[\frac{|\gamma_1'(\theta_1)|^q + |\gamma_1'(\theta_2)|^q}{2} \right]^{\frac{1}{q}} \\ & \times \min \left\{ \int_0^1 \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha, \int_0^1 \int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda) d\lambda d\alpha, \int_0^1 \int_{M(\alpha)}^{V(\alpha)} \gamma_2(\lambda) d\lambda d\alpha \right\}. \end{aligned} \quad (2. 21)$$

Theorem 2.10. *Let $\gamma_1 : \mathcal{U} \subseteq \mathcal{R} \rightarrow \mathcal{R}$ be a differentiable mapping on \mathcal{U}° and $\gamma_2 : [\theta_1, \theta_2] \rightarrow [0, \infty)$ be a continuous and symmetric mapping with respect to $\frac{\theta_1 + \theta_2}{2}$, where $\theta_1, \theta_2 \in \mathcal{U}^\circ$ with $\theta_1 < \theta_2$. If $\gamma_1' \in \mathcal{L}_1([\theta_1, \theta_2])$ and $|\gamma_1'|^q$ is convex on $[\theta_1, \theta_2]$ for $q > 1$, then*

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \left(\frac{\theta_2 - \theta_1}{2\mathbf{m}^{1-\frac{1}{q}}} \right) \left[\int_0^{\frac{\mathbf{m}}{2}} \left(\int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda \right)^{\frac{q}{q-1}} d\alpha \right]^{1-\frac{1}{q}} \\ & \times \left\{ \left(\frac{|\gamma_1'(\theta_1)|^q + 3|\gamma_1'(\theta_2)|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{3|\gamma_1'(\theta_1)|^q + |\gamma_1'(\theta_2)|^q}{8} \right)^{\frac{1}{q}} \right\}, \end{aligned} \quad (2. 22)$$

where \mathbf{m} is a positive integer.

Proof. Using (2. 8) and Hölder integral inequality, we get

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \left(\frac{\theta_2 - \theta_1}{2\mathbf{m}} \right) \left[\int_0^{\frac{\mathbf{m}}{2}} \left(\int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda \right)^{\frac{q}{q-1}} d\alpha \right]^{1-\frac{1}{q}} \\ & \times \left\{ \left(\int_0^{\frac{\mathbf{m}}{2}} |\gamma_1'(\zeta_2(\alpha, \mathbf{m}))|^q d\alpha \right)^{\frac{1}{q}} + \left(\int_0^{\frac{\mathbf{m}}{2}} |\gamma_1'(\zeta_1(\alpha, \mathbf{m}))|^q d\alpha \right)^{\frac{1}{q}} \right\}. \end{aligned} \quad (2. 23)$$

Using the convexity of $|\gamma'_1|^q$ on $[\theta_1, \theta_2]$ for $q \geq 1$, we get

$$\begin{aligned}
& \left(\int_0^{\frac{m}{2}} |\gamma'_1(\zeta_2(\alpha, \mathbf{m}))|^q d\alpha \right)^{\frac{1}{q}} + \left(\int_0^{\frac{m}{2}} |\gamma'_1(\zeta_1(\alpha, \mathbf{m}))|^q d\alpha \right)^{\frac{1}{q}} \\
& \leq \left(\int_0^{\frac{m}{2}} \left[|\gamma'_1(\theta_1)|^q \frac{\alpha}{\mathbf{m}} + \left(\frac{\mathbf{m}-\alpha}{\mathbf{m}} \right) |\gamma'_1(\theta_2)|^q \right] d\alpha \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^{\frac{m}{2}} \left[|\gamma'_1(\theta_2)|^q \frac{\alpha}{\mathbf{m}} + \left(\frac{\mathbf{m}-\alpha}{\mathbf{m}} \right) |\gamma'_1(\theta_1)|^q \right] d\alpha \right)^{\frac{1}{q}} \\
& = \left(\frac{\mathbf{m} |\gamma'_1(\theta_1)|^q + 3\mathbf{m} |\gamma'_1(\theta_2)|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{3\mathbf{m} |\gamma'_1(\theta_1)|^q + \mathbf{m} |\gamma'_1(\theta_2)|^q}{8} \right)^{\frac{1}{q}}. \quad (2.24)
\end{aligned}$$

Combining (2.24) and (2.23), we obtain (2.22). \square

Corollary 2.11. *Suppose that the assumptions of Theorem 2.10 are satisfied and $\mathbf{m} = 2$, then*

$$\begin{aligned}
& \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\
& \leq \left(\frac{\theta_2 - \theta_1}{4} \right) \left[\int_0^1 \left(\int_{\zeta_1(\alpha, 2)}^{\zeta_2(\alpha, 2)} \gamma_2(\lambda) d\lambda \right)^{\frac{q}{q-1}} d\alpha \right]^{1-\frac{1}{q}} \\
& \quad \times \left\{ \left(\frac{|\gamma'_1(\theta_1)|^q + 3|\gamma'_1(\theta_2)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|\gamma'_1(\theta_1)|^q + |\gamma'_1(\theta_2)|^q}{4} \right)^{\frac{1}{q}} \right\}, \quad (2.25)
\end{aligned}$$

where $\zeta_1(\alpha, 2) = \frac{\alpha}{2}\theta_2 + \left(\frac{2-\alpha}{2}\right)\theta_1$, $\zeta_2(\alpha, 2) = \frac{\alpha}{2}\theta_1 + \left(\frac{2-\alpha}{2}\right)\theta_2$.

Corollary 2.12. *Combining (1. 7) and (2. 25), we have*

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \frac{(\theta_2 - \theta_1)}{4} \left\{ \left(\frac{3|\gamma'_1(\theta_1)|^q + |\gamma'_1(\theta_2)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|\gamma'_1(\theta_1)|^q + 3|\gamma'_1(\theta_2)|^q}{4} \right)^{\frac{1}{q}} \right\} \\ & \times \min \left\{ \left[\int_0^1 \left(\int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda \right)^{\frac{q}{q-1}} d\alpha \right]^{1-\frac{1}{q}}, \left[\int_0^1 \left(\int_{T(\alpha)}^{S(\alpha)} \gamma_2(\lambda) d\lambda \right)^{\frac{q}{q-1}} d\alpha \right]^{1-\frac{1}{q}}, \right. \\ & \left. \left[\int_0^1 \left(\int_{M(\alpha)}^{V(\alpha)} \gamma_2(\lambda) d\lambda \right)^{\frac{q}{q-1}} d\alpha \right]^{1-\frac{1}{q}} \right\}. \quad (2. 26) \end{aligned}$$

Theorem 2.13. *Suppose that the assumptions of Theorem 2.5 are satisfied and $|\gamma'_1|$ is quasi-convex on $[\theta_1, \theta_2]$, then the inequality holds:*

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \left(\frac{\theta_2 - \theta_1}{2\mathbf{m}} \right) \left[\max \left\{ |\gamma'_1(\theta_1)|, \left| \gamma'_1 \left(\frac{\theta_2 + \theta_1}{2} \right) \right| \right\} \right. \\ & \left. + \max \left\{ |\gamma'_1(\theta_2)|, \left| \gamma'_1 \left(\frac{\theta_1 + \theta_2}{2} \right) \right| \right\} \right] \int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha, \quad (2. 27) \end{aligned}$$

where $\zeta_1(\alpha, \mathbf{m}), \zeta_2(\alpha, \mathbf{m})$ are defined as in Lemma 2.1 and $\mathbf{m} \geq 1$ is an integer.

Proof. From the identity (2. 8), we have

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \left(\frac{\theta_2 - \theta_1}{2\mathbf{m}} \right) \left[\int_0^{\frac{\mathbf{m}}{2}} \left(\int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda \right) [|\gamma'_1(\zeta_2(\alpha, \mathbf{m}))| + |\gamma'_1(\zeta_1(\alpha, \mathbf{m}))|] d\alpha \right]. \quad (2. 28) \end{aligned}$$

By the quasi-convexity of $|\gamma'_1|$ on $[\theta_1, \theta_2]$, we have

$$|\gamma'_1(\zeta_2(\alpha, \mathbf{m}))| \leq \max \left\{ |\gamma'_1(\theta_2)|, \left| \gamma'_1 \left(\frac{\theta_2 + \theta_1}{2} \right) \right| \right\} \quad (2. 29)$$

and

$$|\gamma'_1(\zeta_2(\alpha, \mathbf{m}))| \leq \max \left\{ |\gamma'_1(\theta_1)|, \left| \gamma'_1 \left(\frac{\theta_1 + \theta_2}{2} \right) \right| \right\} \quad (2.30)$$

for all $\alpha \in [0, \frac{\mathbf{m}}{2}]$.

Combining the inequalities in (2.28), (2.29) and (2.30), we obtain the inequality (2.27). \square

Remark 2.14. If $|\gamma'_1|$ is non-decreasing in Theorem 2.13, then

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \left(\frac{\theta_2 - \theta_1}{2\mathbf{m}} \right) \left[\left(|\gamma'_1(\theta_2)| + \left| \gamma'_1 \left(\frac{\theta_2 + \theta_1}{2} \right) \right| \right) \int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \right] \quad (2.31) \end{aligned}$$

and if $|\gamma'_1|$ is non-increasing in Theorem 2.13, then

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \left(\frac{\theta_2 - \theta_1}{2\mathbf{m}} \right) \left[\left(|\gamma'_1(\theta_1)| + \left| \gamma'_1 \left(\frac{\theta_2 + \theta_1}{2} \right) \right| \right) \int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \right]. \quad (2.32) \end{aligned}$$

Theorem 2.15. Suppose that the assumptions of Theorem 2.5 are satisfied and $|\gamma'_1|^{\mathbf{q}}$ is quasi-convex on $[\theta_1, \theta_2]$ for $\mathbf{q} \geq 1$, then the inequality holds:

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \left(\frac{\theta_2 - \theta_1}{\mathbf{m}^{1-\frac{1}{\mathbf{q}}}} \right) \left(\frac{1}{2} \right)^{\frac{1}{\mathbf{q}}+1} \left[\left(\max \left\{ |\gamma'_1(\theta_1)|^{\mathbf{q}}, \left| \gamma'_1 \left(\frac{\theta_2 + \theta_1}{2} \right) \right|^{\mathbf{q}} \right\} \right)^{\frac{1}{\mathbf{q}}} \right. \\ & \quad \left. + \left(\max \left\{ |\gamma'_1(\theta_2)|^{\mathbf{q}}, \left| \gamma'_1 \left(\frac{\theta_1 + \theta_2}{2} \right) \right|^{\mathbf{q}} \right\} \right)^{\frac{1}{\mathbf{q}}} \right] \int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha, \quad (2.33) \end{aligned}$$

where $\zeta_1(\alpha, \mathbf{m})$, $\zeta_2(\alpha, \mathbf{m})$ are defined as in Lemma 2.1 and $\mathbf{m} \geq 1$ is an integer.

Proof. Using the identity (2. 8) and the power-mean integral inequality, we get

$$\begin{aligned}
& \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\
& \leq \left(\frac{\theta_2 - \theta_1}{2\mathbf{m}} \right) \left[\int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \right]^{1 - \frac{1}{\mathbf{q}}} \\
& \times \left\{ \left(\int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \int_0^{\frac{\mathbf{m}}{2}} |\gamma_1'(\zeta_2(\alpha, \mathbf{m}))|^{\mathbf{q}} d\alpha \right)^{\frac{1}{\mathbf{q}}} \right. \\
& \quad \left. + \left(\int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \int_0^{\frac{\mathbf{m}}{2}} |\gamma_1'(\zeta_1(\alpha, \mathbf{m}))|^{\mathbf{q}} d\alpha \right)^{\frac{1}{\mathbf{q}}} \right\}. \quad (2. 34)
\end{aligned}$$

By the quasi-convexity of $|\gamma_1'|^{\mathbf{q}}$ on $[\theta_1, \theta_2]$ for $\mathbf{q} \geq 1$, we have

$$|\gamma_1'(\zeta_2(\alpha, \mathbf{m}))|^{\mathbf{q}} \leq \max \left\{ |\gamma_1'(\theta_1)|^{\mathbf{q}}, \left| \gamma_1' \left(\frac{\theta_2 + \theta_1}{2} \right) \right|^{\mathbf{q}} \right\} \quad (2. 35)$$

and

$$|\gamma_1'(\zeta_1(\alpha, \mathbf{m}))|^{\mathbf{q}} \leq \max \left\{ |\gamma_1'(\theta_2)|^{\mathbf{q}}, \left| \gamma_1' \left(\frac{\theta_1 + \theta_2}{2} \right) \right|^{\mathbf{q}} \right\} \quad (2. 36)$$

for all $\alpha \in [0, \frac{\mathbf{m}}{2}]$.

Combining the inequalities (2. 34), (2. 35) and (2. 36), we obtain the inequality (2. 33). \square

Remark 2.16. If $|\gamma_1'|$ is non-decreasing in Theorem 2.15, then

$$\begin{aligned}
& \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\
& \leq \left(\frac{\theta_2 - \theta_1}{\mathbf{m}^{1 - \frac{1}{\mathbf{q}}}} \right) \left(\frac{1}{2} \right)^{\frac{1}{\mathbf{q}} + 1} \left[\left(|\gamma_1'(\theta_2)| + \left| \gamma_1' \left(\frac{\theta_2 + \theta_1}{2} \right) \right| \right) \right] \int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha \quad (2. 37)
\end{aligned}$$

and if $|\gamma_1'|$ is non-increasing in Theorem 2.15, then

$$\begin{aligned} & \left| \frac{\gamma_1(\theta_1) + \gamma_1(\theta_2)}{2} \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) d\lambda - \int_{\theta_1}^{\theta_2} \gamma_2(\lambda) \gamma_1(\lambda) d\lambda \right| \\ & \leq \left(\frac{\theta_2 - \theta_1}{\mathbf{m}^{1-\frac{1}{\mathbf{q}}}} \right) \left(\frac{1}{2} \right)^{\frac{1}{\mathbf{q}}+1} \left[\left(|\gamma_1'(\theta_1)| + \left| \gamma_1' \left(\frac{\theta_2 + \theta_1}{2} \right) \right| \right) \right] \int_0^{\frac{\mathbf{m}}{2}} \int_{\zeta_1(\alpha, \mathbf{m})}^{\zeta_2(\alpha, \mathbf{m})} \gamma_2(\lambda) d\lambda d\alpha. \end{aligned} \quad (2.38)$$

Remark 2.17. A number of interesting results can be obtained from our results for $\mathbf{m} > 2$.

3. APPLICATIONS TO SPECIAL MEANS

For positive numbers $\theta_1 > 0$ and $\theta_2 > 0$, define

$$A(\theta_1, \theta_2) = \frac{\theta_1 + \theta_2}{2},$$

$$G(\theta_1, \theta_2) = \sqrt{\theta_1 \theta_2}$$

and

$$L_r(\theta_1, \theta_2) = \begin{cases} \left[\frac{\theta_2^{r+1} - \theta_1^{r+1}}{(r+1)(\theta_2 - \theta_1)} \right]^{\frac{1}{r}}, & r \neq -1, 0, \\ \frac{\theta_2 - \theta_1}{\ln \theta_2 - \ln \theta_1}, & r = -1, \\ \frac{1}{e} \left(\frac{\theta_2^{\theta_2}}{\theta_1^{\theta_1}} \right)^{\frac{1}{\theta_2 - \theta_1}}, & r = 0. \end{cases}$$

$A(\theta_1, \theta_2)$, $G(\theta_1, \theta_2)$ and $L_r(\theta_1, \theta_2)$ are called the arithmetic, geometric mean and generalized logarithmic means respectively of θ_1 and θ_2 .

Let

$$\gamma_1(\lambda) = \frac{\mathbf{q}\lambda^{1+\frac{1}{\mathbf{q}}}}{\mathbf{q}+1} \text{ for } \lambda > 0, \mathbf{q} \geq 1. \quad (3.39)$$

Then, obviously

$$|\gamma_1'(\lambda)|^{\mathbf{q}} = \lambda$$

is convex on $[\theta_1, \theta_2]$.

Moreover, the function

$$\gamma_2(\lambda) = \left(\lambda - \frac{\theta_1 + \theta_2}{2} \right)^2, \quad (3.40)$$

where $\theta_1, \theta_2 > 0$ and $\lambda \in [\theta_1, \theta_2]$, is a symmetric mapping with respect to $\frac{\theta_1 + \theta_2}{2}$ on $[\theta_1, \theta_2]$.

Considering the functions (3.39) and (3.40), we have the following inequalities of special means $A(\theta_1, \theta_2)$, $G(\theta_1, \theta_2)$ and $L_r(\theta_1, \theta_2)$ using Theorem 2.5 and Theorem 2.10.

Theorem 3.1. *If $\theta_2 > \theta_1 > 0$ and $q \geq 1$, then*

$$\left| \frac{q(\theta_2 - \theta_1)^2 A\left(\theta_1^{1+\frac{1}{q}}, \theta_2^{1+\frac{1}{q}}\right)}{6(q+1)} - \frac{q(4q^2 + 3q + 1) L_{3+\frac{1}{q}}^{3+\frac{1}{q}}(\theta_1, \theta_2)}{2(q+1)(2q+1)(3q+1)} \right. \\ \left. + \frac{qG^2(\theta_1, \theta_2) L_{1+\frac{1}{q}}^{1+\frac{1}{q}}(\theta_1, \theta_2)}{(3q+1)} - \frac{qG^4(\theta_1, \theta_2) L_{\frac{1}{q}-1}^{\frac{1}{q}-1}(\theta_1, \theta_2)}{2(q+1)(2q+1)} \right| \\ \leq \left[\frac{(\theta_2 - \theta_1)^3}{48} \right]^{1-\frac{1}{q}} A^{\frac{1}{q}}(\theta_1, \theta_2) \quad (3.41)$$

and if $q > 1$, then

$$\left| \frac{q(\theta_2 - \theta_1)^2 A\left(\theta_1^{1+\frac{1}{q}}, \theta_2^{1+\frac{1}{q}}\right)}{6(q+1)} - \frac{q(4q^2 + 3q + 1) L_{3+\frac{1}{q}}^{3+\frac{1}{q}}(\theta_1, \theta_2)}{2(q+1)(2q+1)(3q+1)} \right. \\ \left. + \frac{qG^2(\theta_1, \theta_2) L_{1+\frac{1}{q}}^{1+\frac{1}{q}}(\theta_1, \theta_2)}{(3q+1)} - \frac{qG^4(\theta_1, \theta_2) L_{\frac{1}{q}-1}^{\frac{1}{q}-1}(\theta_1, \theta_2)}{2(q+1)(2q+1)} \right| \\ \leq \frac{(b-a)^3}{24} \left(\frac{q-1}{4q-1} \right)^{1-\frac{1}{q}} \left\{ \left(\frac{\theta_1 + 3\theta_2}{8} \right)^{\frac{1}{q}} + \left(\frac{3\theta_1 + \theta_2}{8} \right)^{\frac{1}{q}} \right\}. \quad (3.42)$$

Corollary 3.2. *If the hypotheses of Theorem 3.1 are satisfied and if $q = 1$, then*

$$\left| \frac{(\theta_2 - \theta_1)^2 A(\theta_1^2, \theta_2^2)}{12} - \frac{L_4^4(\theta_1, \theta_2)}{6} \right. \\ \left. + \frac{G^2(\theta_1, \theta_2) L_2^2(\theta_1, \theta_2)}{4} - \frac{G^4(\theta_1, \theta_2)}{12} \right| \leq A(\theta_1, \theta_2) \quad (3.43)$$

4. CONCLUSIONS

In this paper, the results are based on a new family of identities for a positive integer m . We have established new family of Fejér and Hermite-Hadamard type inequalities for the functions whose derivatives satisfy assumptions of convexity and quasi-convexity based on the new family of identities for a positive integer m . The ideas and methods to acquire the studies are expected to inspire the interested readers. We suspect that our findings can be extended to obtain various results in convex analysis, special functions, theories related to optimization, mathematical inequalities and can invigorate further research work in various fields of pure and applied sciences.

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