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ES Structure Based on Soft J-Subset

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Abstract: The ES structure described by soft subsets or soft M-subsets does not yield a lattice structure due to its restriction on parameter sets, and so cannot be used in information theory. This study proposes a new ES structure on soft sets that addresses the deficiencies of the prior structure. Using mathematical concepts, we can construct and entirely new system of soft sets. As a result, the ES structure is derived from a finite collection of basic soft sets and offers complicated soft sets via its ES components, allowing for it to be operated by computers, as this is more acceptable to conventional mathematical viewpoints. We rewrote this using a soft J-subset and demonstrated that (ES, $\tilde{\vee}_{ES}$, $\tilde{\wedge}_{ES}$) is a distributive lattice. This will play an important role in decision-making problems and contribute to a better understanding of human recognition processes. During the process of reaching a decision, several groups of parameters develop, and the ES structure in this article takes these parameters into consideration in order to handle the intricate issues that arise. In soft set theory, this research gives insight into the cognitive field.

Keywords: soft sets; soft M-subset; soft J-subset; lattice; ES structure

MSC: 06D72; 03E20; 03E72

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1. Introduction

We can view three areas as mathematical methods to cope with complexity in mathematics: probability theory, fuzzy sets, and interval mathematics. Zadeh [1] has shown that one of the most relevant theories for coping with uncertainty is the fuzzy set theory. However, as Molodtsov [2] noted, each of these theories includes inherent deficiencies. They [2,3] hypothesized that one reason for these difficulties could be a gap in the parameterization approach. Molodtsov proposed the soft set idea as an innovative mathematical method of coping with ambiguity in order to address these issues. A soft set is a mapping that assigns one crisp subset of the universe set to each attribute/parameter. The difficulty of defining the membership function, and other connected issues basically does not exist in soft set theory. Because of this, the theory is incredibly practical and has possible applications in different branches of mathematics. Maji et al. and Nasef et al. [4,5] solved a decision-making problem in soft set theory by utilizing reduct-soft sets. Acharjee et al. [6] reviewed soft set theory connections and hybrid structures in a non-technical manner, and focused on two important questions: 1. What is the future of a hybrid mathematical soft set structure in science and social science? 2. Why should we take care to use hybrid soft set structures? To find solutions to decision-making problems, we create some representations using soft set theory, such as Type-2 Fuzzy soft set representation, which is the simplest form used to extend soft set theory to capture more uncertainty and was given by Zhang [7]. It has to be proved that a group of soft sets with new operations creates a structure of soft sets and that soft sets with a given set of attributes include MV algebra and BCK algebra.

In [8,9], the author Liu suggested an infinite distributive molecular lattice using fuzzy sets and termed it El-algebra. In addition, they proposed a revolutionary system "AFS

Structure" of fuzzy sets and systems that is more acceptable than the conventional mathematical perspectives. Motivated by this, Yadav and Singh investigated El-algebra in soft sets, and suggested the term soft El-algebra and some remarkable algebraic properties [10]. Further, in [11], we offered a thorough and comprehensive survey of the research on soft set theory and the developments of topological spaces in soft sets.

The idea of soft subsets was initially described in a very precise way by Maji et al. [3]. They also have written a thorough theoretical investigation of soft sets and suggested a few outcomes involving soft distributive laws in relation to $\tilde{\wedge}$-product and $\tilde{\vee}$-product operations of soft sets, but provided no evidence to support their claims. Furthermore, Ali et al. [12] noted that the findings in [3] are not valid. Therefore, Jun and Yang put forward the ideas of generalized soft subsets [13]. They also attempted to use generalized soft equal relations to address Maji's findings [3] and proposed generalized soft distribution laws. Finally, they defined soft J-equal relations and soft L-equal relations. Jun and Yang in [13,14] concluded that not all types of soft equal relations are subject to distributive laws. Furthermore, Feng and Li [15] provided a systematic study of several types of soft subsets, and some incomplete results concerning soft product operations, as well as investigating their algebraic properties. Ma et al. [16] also pointed out a defect—that the definitions of soft subset and soft intersection in Maji's paper are partial definitions—and then intoduced an extension of soft set to simplify operations between soft sets.

As was covered in the above paragraph, the join and meet operations for the distributive lattice do not operate particularly well. This served as a driving force behind our decision to conduct research on those soft subsets that have the required structure for the algorithm to be successfully implemented. This roadblock is removed with the help of the soft J subset. In the previous paper [17], the ES structure described by soft subsets or soft M-subsets does not yield a lattice structure due to their restriction on parameter sets. As a result, the research demonstrates another way of using soft J-subsets, and we form the ES structure as a distributive lattice in this work. This research provides a structure that extends soft sets to obtain a more exact understanding of the cognitive field. It investigates the lattice structure (ES, $\tilde{\vee}_{ES}$, $\tilde{\wedge}_{ES}$) in depth. Firstly, we provide some basic definitions of soft sets and their operations with some important theorems and results along, with examples. Then, we update the ES- structure by applying soft J-subsets, which are generated by relaxing the restrictions on the parameter set. In addition, we amend and rectify a few results in the theorem, results and features of the ES structure presented in [17]. Lastly, we provide two examples with an algorithm which shows how to solve complex problems in decision-making.

2. Preliminaries

This section provides some basic definitions of soft sets and their operators, with some results illustrated by examples. We will utilize these definitions to construct our structure, mainly the soft J-subset, "AND" and "OR" operators of soft sets. Here, let P($\mathfrak{U}_s$) be the power set of universe set, $\mathfrak{U}_s$, and $\mathcal{E}$ be the set of all parameters/attributes defined over $\mathfrak{U}_S$. Then,

Definition 1 ([2])**.** *A couple* $(\mathfrak{F}, \mathcal{A})$*,* $\mathcal{A} \subseteq \mathcal{E}$*, written as* $\mathfrak{F}_{\mathcal{A}}$*, is known to be soft set over* $\mathfrak{U}_s$*, if* $\mathfrak{F}$ *is a representation defined as:*

$$\mathfrak{F} : \mathcal{A} \longrightarrow P(\mathfrak{U}_s).$$

where $\mathfrak{F}_{\mathcal{A}} = \{(\alpha_i, \mathfrak{F}(\alpha_i)) \mid \alpha_i \in \mathcal{A}, \mathfrak{F}(\alpha_i) \in P(\mathfrak{U}_s)\}$*. The value of* $\mathfrak{F}(\alpha_i)$ *can be chosen at random.* $\mathfrak{F}_{\mathcal{A}}$ *is not a normal or classical set. A lot of data were generated in [2].*

Definition 2 ([3])**.** *The conjunction of two soft sets is defined by "AND" (*$\tilde{\wedge}$*) operator as:* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\wedge} \mathfrak{F}^2_{\mathcal{A}_2} = \mathfrak{F}_{\mathcal{A}_1 \times \mathcal{A}_2}$*,* $(\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{E})$*, where* $\mathfrak{F}(\alpha_i, \alpha_j) = \mathfrak{F}^1(\alpha_i) \cap \mathfrak{F}^2(\alpha_j)$ *for all* $(\alpha_i, \alpha_j) \in \mathcal{A}_1 \times \mathcal{A}_2$*.*

Definition 3 ([3]). *The disjunction of two soft sets is defined by "OR" ($\tilde{\vee}$) operator as:* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\vee} \mathfrak{F}^2_{\mathcal{A}_2} = \mathfrak{F}_{\mathcal{A}_1 \times \mathcal{A}_2}$, *(*$\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{E}$*), where* $\mathfrak{F}(\alpha_i, \alpha_j) = \mathfrak{F}^1(\alpha_i) \cup \mathfrak{F}^2(\alpha_j)$ *for all* $(\alpha_i, \alpha_j) \in \mathcal{A}_1 \times \mathcal{A}_2$.

Definition 4 ([3]). *The soft set* $\mathfrak{F}^1_{\mathcal{A}_1}$ *is a soft subset or soft M-subset of* $\mathfrak{F}^2_{\mathcal{A}_2}$*, represented by* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\subseteq} \mathfrak{F}^2_{\mathcal{A}_2}$ *or* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\subseteq}_M \mathfrak{F}^2_{\mathcal{A}_2}$*, if it fulfils the following requirements:*

(i) $\mathcal{A}_1 \subseteq \mathcal{A}_2$,

(ii) *For each* $a_1 \in \mathcal{A}_1$, $\mathfrak{F}^1(a_1)$ *and* $\mathfrak{F}^2(a_1)$ *are similar approximations.*

$\mathfrak{F}^1_{\mathcal{A}_1}$ *and* $\mathfrak{F}^2_{\mathcal{A}_2}$ *are soft equal or soft M-equal if, and only if,* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\subseteq}_M \mathfrak{F}^2_{\mathcal{A}_2}$ *and* $\mathfrak{F}^2_{\mathcal{A}_2} \tilde{\subseteq}_M \mathfrak{F}^1_{\mathcal{A}_1}$.

Definition 5 ([14]). *The soft set* $\mathfrak{F}^1_{\mathcal{A}_1}$ *is said to be soft J-subset of* $\mathfrak{F}^2_{\mathcal{A}_2}$*, written by* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\subseteq}_J \mathfrak{F}^2_{\mathcal{A}_2}$*, if, and only if, for any* $a_1 \in \mathcal{A}_1$, $\exists\, a_2 \in \mathcal{A}_2$ *such that* $\mathfrak{F}^1(a_1) \subseteq \mathfrak{F}^2(a_2)$ *(see Example 1). Moreover,* $\mathfrak{F}^1_{\mathcal{A}_1}$ *and* $\mathfrak{F}^2_{\mathcal{A}_2}$ *are called soft J-equal, denoted as* $\mathfrak{F}^1_{\mathcal{A}_1} =_J \mathfrak{F}^2_{\mathcal{A}_2}$*, if* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\subseteq}_J \mathfrak{F}^2_{\mathcal{A}_2}$ *and* $\mathfrak{F}^2_{\mathcal{A}_2} \tilde{\subseteq}_J \mathfrak{F}^1_{\mathcal{A}_1}$.

Example 1. *Consider a universe set* $\mathfrak{U}_s = \{u_1, u_2, u_3\}$ *and attribute set* $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$*. Let* $\mathfrak{F}^1_{\mathcal{A}_1} = \{(e_1, \{u_1, u_2\}), (e_2, \{u_3\}), (e_3, \{u_2, u_3\})\}$ *and* $\mathfrak{F}^2_{\mathcal{A}_2} = \{(e_1, \{u_1, u_2\}), (e_3, \{u_2, u_3\}), (e_4, \{u_2\})\}$ *are non-null soft sets over* $\mathfrak{U}_s$*. Since* $\mathcal{A}_1 \neq \mathcal{A}_2$, $\mathfrak{F}^1_{\mathcal{A}_1}$ *is not soft equal or soft M-equal to* $\mathfrak{F}^2_{\mathcal{A}_2}$*. However, we can see that* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\subseteq}_J \mathfrak{F}^2_{\mathcal{A}_2}$ *and* $\mathfrak{F}^2_{\mathcal{A}_2} \tilde{\subseteq}_J \mathfrak{F}^1_{\mathcal{A}_1}$*. Hence,* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{=}_J \mathfrak{F}^2_{\mathcal{A}_2}$.

Theorem 1. *If* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\subseteq}_J \mathfrak{F}^2_{\mathcal{A}_2}$ *and* $\mathfrak{F}^3_{\mathcal{A}_3} \tilde{\subseteq}_J \mathfrak{F}^4_{\mathcal{A}_4}$*, then* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\vee} \mathfrak{F}^3_{\mathcal{A}_3} \tilde{\subseteq}_J \mathfrak{F}^2_{\mathcal{A}_2} \tilde{\vee} \mathfrak{F}^4_{\mathcal{A}_4}$ *and* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\wedge} \mathfrak{F}^3_{\mathcal{A}_3} \tilde{\subseteq}_J \mathfrak{F}^2_{\mathcal{A}_2} \tilde{\wedge} \mathfrak{F}^4_{\mathcal{A}_4}$.

Proof. Therefore, $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\subseteq}_J \mathfrak{F}^2_{\mathcal{A}_2}$, so for every $\alpha_i \in \mathcal{A}_1$, $\exists\, \alpha_j \in \mathcal{A}_2$, such that $\mathfrak{F}^1(\alpha_i) \subseteq \mathfrak{F}^2(\alpha_j)$. Similarly, for every $\alpha_k \in \mathcal{A}_3$, $\exists\, \alpha_l \in \mathcal{A}_4$, such that $\mathfrak{F}^3(\alpha_k) \subseteq \mathfrak{F}^4(\alpha_l)$. Let $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\vee} \mathfrak{F}^3_{\mathcal{A}_3} = \mathfrak{F}_{\mathcal{A}_1 \times \mathcal{A}_3}$ and $\mathfrak{F}^2_{\mathcal{A}_2} \tilde{\vee} \mathfrak{F}^4_{\mathcal{A}_4} = \mathfrak{F}_{\mathcal{A}_2 \times \mathcal{A}_4}$. Then, $\mathfrak{F}(\alpha_i, \alpha_k) = \mathfrak{F}^1(\alpha_i) \cup \mathfrak{F}^3(\alpha_k)$, $\forall\, (\alpha_i, \alpha_k) \in \mathcal{A}_1 \times \mathcal{A}_3$. But $\mathfrak{F}^1(\alpha_i) \cup \mathfrak{F}^3(\alpha_k) \subseteq \mathfrak{F}^2(\alpha_j) \cup \mathfrak{F}^4(\alpha_l)$. This implies that, for any $(\alpha_i, \alpha_k) \in \mathcal{A}_1 \times \mathcal{A}_3$, $\exists\, (\alpha_j, \alpha_l) \in \mathcal{A}_2 \times \mathcal{A}_4$ such that $\mathfrak{F}^1(\alpha_i) \cup \mathfrak{F}^3(\alpha_k) \subseteq \mathfrak{F}^2(\alpha_j) \cup \mathfrak{F}^4(\alpha_l)$. Hence, from Definition 5, $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\vee} \mathfrak{F}^3_{\mathcal{A}_3} \tilde{\subseteq}_J \mathfrak{F}^2_{\mathcal{A}_2} \tilde{\vee} \mathfrak{F}^4_{\mathcal{A}_4}$. Similarly, we have $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\wedge} \mathfrak{F}^3_{\mathcal{A}_3} \tilde{\subseteq}_J \mathfrak{F}^2_{\mathcal{A}_2} \tilde{\wedge} \mathfrak{F}^4_{\mathcal{A}_4}$. □

Result 1. *Let* $\mathfrak{F}^1_{\mathcal{A}_1}$ *and* $\mathfrak{F}^2_{\mathcal{A}_2}$ *are two non-null soft sets. Then,* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\wedge} \mathfrak{F}^2_{\mathcal{A}_2}$ *and* $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\cap} \mathfrak{F}^2_{\mathcal{A}_2}$ *can be a null soft set (see Examples 2 and 3) i.e., if* $\mathfrak{F}^1_{\mathcal{A}_1}, \mathfrak{F}^2_{\mathcal{A}_2} \neq \phi$*, then* $\forall\, \alpha_i \in \mathcal{A}_1, \alpha_j \in \mathcal{A}_2$ *such that* $\mathfrak{F}^1(\alpha_i) \cap \mathfrak{F}^2(\alpha_j) = \phi$.

Example 2. *Let* $\mathfrak{F}^1_{\mathcal{A}_1} = \{(\alpha_1, \{u_1, u_2\}), (\alpha_2, \{u_5\})\}$ *and* $\mathfrak{F}^2_{\mathcal{A}_2} = \{(\alpha_1, \{u_3\}), (\alpha_3, \{u_5\})\}$ *are non-null soft sets over* $\mathfrak{U}_s = \{u_1, u_2, u_3, u_4, u_5\}$*. Since* $\mathcal{A}_1 \times \mathcal{A}_2 = \{(\alpha_1, \alpha_1), (\alpha_1, \alpha_3), (\alpha_2, \alpha_3), (\alpha_2, \alpha_1)\}$*. Thus,*

$\mathfrak{F}^3_{\mathcal{A}_3} = \mathfrak{F}^1_{\mathcal{A}_1} \tilde{\wedge} \mathfrak{F}^2_{\mathcal{A}_2} = \{((\alpha_i, \alpha_j), \phi) \mid (\alpha_i, \alpha_j) \in \mathcal{A}_1 \times \mathcal{A}_2\}$.

Example 3. *Consider* $\mathfrak{F}^1_{\mathcal{A}_1}$ *and* $\mathfrak{F}^2_{\mathcal{A}_2}$ *as soft sets over* $\mathfrak{U}_s = \{u_i \mid i \in \{1, 2, 3, 4, 5\}\}$*, defined as:*

$\mathfrak{F}^1_{\mathcal{A}_1} = \{(\alpha_1, \{u_1, u_2\}), (\alpha_2, \{u_4, u_5\})\}$

$\mathfrak{F}^2_{\mathcal{A}_2} = \{(\alpha_1, \{u_3\}), (\alpha_2, \{u_1, u_2\}), (\alpha_3, \{u_4, u_5\})\}$

Let $\mathfrak{F}^1_{\mathcal{A}_1} \tilde{\cap} \mathfrak{F}^2_{\mathcal{A}_2} = \mathfrak{F}^3_{\mathcal{A}_3}$*. Now* $\mathcal{A}_3 = \mathcal{A}_1 \cap \mathcal{A}_2 = \{\alpha_1, \alpha_2\}$*. Hence,* $\mathfrak{F}^3_{\mathcal{A}_3} = \{(\alpha_1, \phi), (\alpha_2, \phi)\}$.

Here, we will first revise the ES structure of soft sets [17] and make some changes to the propositions and theorems by utilizing *Soft J-subset* [13,14] and some results. Moreover, we will prove that the relation $\mathcal{R}$ on the ES structure is an equivalence relation. However, their proofs will remain the same as in [17], so we can not explain this again in this paper.

3. Revised ES Structure

In the previous paper, an ES structure was constructed using all soft sets, $\mathfrak{F}_{\mathcal{A}} \in \mathcal{S}$, defined over a common universe $\mathfrak{U}_s$, and a complete lattice was formed utilizing the soft M-subset. However, due to restrictions on the parameter set in the soft M-subset, it can not form a complete lattice. The soft J-subset was found to be the best replacement of the soft M-subset, satisfying all the criteria. The soft J-subset has no such restriction on the parameter set, and hence it is easy to use to define binary operations on ES. This article used the following methodology:

1. Firstly, take all soft sets described over a common universe $\mathfrak{U}_s$ as:
$$\mathcal{S} = \{\mathfrak{F}_{\mathcal{A}} \mid \mathfrak{F} : \mathcal{A} \longrightarrow P(\mathfrak{U}_s), \mathcal{A} \subseteq \mathcal{E}\}$$
2. Construct all ES components using the operators' conjunction ($\tilde{\wedge}$) and disjunction ($\tilde{\vee}$) as follows:
$$ES = \{\textstyle\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}}) \mid \mathcal{S}_i \subseteq \mathcal{S}\},$$
where $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}}$ denotes conjunctions ($\tilde{\wedge}$) of soft sets in $\mathcal{S}_i$ and $\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}})$ denotes disjunctions ($\tilde{\vee}$) of the soft sets $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}}$.
3. Define a binary relation $\mathcal{R}$ on ES by utilizing the soft J-subset and prove it to be an equivalence relation.
4. Define two new binary operations, $\tilde{\wedge}_{ES}$ and $\tilde{\vee}_{ES}$, on ES to form the structure (ES, $\tilde{\wedge}_{ES}$, $\tilde{\vee}_{ES}$), a lattice.
5. Lastly, using the order relation "$\leq_{ES}$", (ES, $\leq_{ES}$) makes a distributive lattice.

Let $\mathcal{S}$ be a set of all possible soft sets over $\mathfrak{U}_s$; many soft sets can be represented by the ES components and the soft logic operations can be performed by $\tilde{\vee}_{ES}$ and $\tilde{\wedge}_{ES}$ in the lattice system (ES, $\tilde{\vee}_{ES}$, $\tilde{\wedge}_{ES}$). As a result, the use of complicated soft sets in ES will reduce the complexity in decision-making. Let us show that the complexity of human conceptions is the direct outcome of some simple soft sets being combined.

In [17], we defined a binary relation $\mathcal{R}$ on ES as: $\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}})\ \mathcal{R}\ \sum_{j\in J}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j} \mathfrak{F}_{\mathcal{A}})$ $\iff$ for any $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}}$, $\exists \prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_h} \mathfrak{F}_{\mathcal{A}}$, such that $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}} \tilde{\subseteq}_M \prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_h} \mathfrak{F}_{\mathcal{A}}$ or $\mathcal{S}_i \supseteq \mathcal{T}_h$. However, if we take $\mathcal{T}_h \subset \mathcal{S}_i$, then the parameter set of $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}}$ cannot make a subset of the parameter set of $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_h} \mathfrak{F}_{\mathcal{A}}$. In this way, we cannot define the relation $\mathcal{R}$ on ES with respect to the soft M-subset. Therefore, here, we revise and correct this by utilizing soft J-subsets and making a distributive lattice (ES, $\tilde{\vee}_{ES}$, $\tilde{\wedge}_{ES}$).

Definition 6. *Let $\mathfrak{F}_{\mathcal{A}}$ be a soft set and $\mathcal{S}$ be the set of soft sets over $\mathfrak{U}_s$. Then, from [17] ES = $\{\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}}) \mid \mathcal{S}_i \subseteq \mathcal{S}\}$. Here, we use soft J-subsets to define a binary relation $\mathcal{R}$ on the ES structure as: for any $\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}})$, $\sum_{j\in J}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j} \mathfrak{F}_{\mathcal{A}}) \in$ ES,*
$\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}})\ \mathcal{R}\ \sum_{j\in J}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j} \mathfrak{F}_{\mathcal{A}}) \iff$

1. *For any $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}}$ ($i \in I$), $\exists \prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_k} \mathfrak{F}_{\mathcal{A}}$ ($k \in J$), such that $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}} \tilde{\subseteq}_J \prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_k} \mathfrak{F}_{\mathcal{A}}$, and*
2. *For any $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j} \mathfrak{F}_{\mathcal{A}}$ ($j \in J$), $\exists \prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_m} \mathfrak{F}_{\mathcal{A}}$ ($m \in I$), such that $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j} \mathfrak{F}_{\mathcal{A}} \tilde{\subseteq}_J \prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_m} \mathfrak{F}_{\mathcal{A}}$.*

Example 4. *Let $\mathfrak{U}_s = \{u_i \mid i \in \{1, 2, ..., 5\}\}$ and $\mathcal{S} = \{\mathfrak{F}^1_{\mathcal{A}}, \mathfrak{F}^2_{\mathcal{B}}, \mathfrak{F}^3_{\mathcal{C}}\}$ be a set of soft sets defined on $\mathfrak{U}_s$ as:*
$\mathfrak{F}^1_{\mathcal{A}} = \{(a_1, \{u_1, u_2\}), (a_2, \{u_2, u_5\})\}$,
$\mathfrak{F}^2_{\mathcal{B}} = \{(b_1, \{u_2, u_3\}), (b_2, \{u_1, u_4\}), (b_3, \{u_5\})\}$,
$\mathfrak{F}^3_{\mathcal{C}} = \{(c_1, \{u_1, u_2, u_5\}), (c_2, \{u_2, u_3, u_4\}), (c_3, \{u_1, u_2, u_4\})\}$.

Let $\mathcal{S}_1 = \{\mathfrak{F}^1_{\mathcal{A}}, \mathfrak{F}^2_{\mathcal{B}}\}$, $\mathcal{S}_2 = \{\mathfrak{F}^3_{\mathcal{C}}\}$, $\mathcal{S}_3 = \{\mathfrak{F}^1_{\mathcal{A}}\}$, $\mathcal{S}_4 = \{\mathfrak{F}^2_{\mathcal{B}}, \mathfrak{F}^3_{\mathcal{C}}\}$ and $\mathcal{S}_5 = \{\mathfrak{F}^2_{\mathcal{B}}\}$. Consider $\sum_{i\in\{1,2,5\}}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i} \mathfrak{F}_{\mathcal{A}})$, $\sum_{j\in\{2,3,4\}}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_j} \mathfrak{F}_{\mathcal{A}}) \in$ ES. Then, $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_1} \mathfrak{F}_{\mathcal{A}} = \mathfrak{F}^1_{\mathcal{A}} \tilde{\wedge} \mathfrak{F}^2_{\mathcal{B}} = \{((a_1,$

b_1), $\{u_2\}$), ((a_1, b_2), $\{u_1\}$), ((a_1, b_3), ϕ), ((a_2, b_1), $\{u_2\}$), ((a_2, b_2), ϕ), ((a_2, b_3), $\{u_5\}$)$\}$ and $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_4}\mathfrak{F}_\mathcal{A}$ = $\{$((b_1, c_1), $\{u_2\}$), ((b_1, c_2), $\{u_2, u_3\}$), ((b_2, c_1), $\{u_1\}$), ((b_2, c_2), $\{u_4\}$), ((b_3, c_1), $\{u_5\}$), ((b_3, c_2), ϕ)$\}$.

We can see that $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_1}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_3}\mathfrak{F}_\mathcal{A}$ but $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_1}\mathfrak{F}_\mathcal{A} \tilde{\nsubseteq}_M \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_3}\mathfrak{F}_\mathcal{A}$, as $\mathcal{A}\times\mathcal{B} \nsubseteq \mathcal{A}$. Additionally, $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_2}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_2}\mathfrak{F}_\mathcal{A}$, $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_3}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_2}\mathfrak{F}_\mathcal{A}$, $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_4}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_5}\mathfrak{F}_\mathcal{A}$ and $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_5}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_2}\mathfrak{F}_\mathcal{A}$. Hence, from Definition 6, we have

$$\textstyle\sum_{i\in\{1,2,5\}}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_i}\mathfrak{F}_\mathcal{A})\ \mathcal{R}\ \sum_{j\in\{2,3,4\}}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A}).$$

Theorem 2. *$\mathcal{R}$ is an equivalence relation on the ES structure.*

Proof. It is trivial to verify that $\mathcal{R}$ is reflexive and symmetric. Therefore, we have to show that $\mathcal{R}$ is transitive.

(i) Let $\sum_{i\in I}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_i}\mathfrak{F}_\mathcal{A})\ \mathcal{R}\ \sum_{j\in J}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A})$ and $\sum_{j\in J}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A})\ \mathcal{R}\ \sum_{k\in K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{U}_k}\mathfrak{F}_\mathcal{A})$. Since, $\sum_{i\in I}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_i}\mathfrak{F}_\mathcal{A})\ \mathcal{R}\ \sum_{j\in J}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A})$, so for every $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_i}\mathfrak{F}_\mathcal{A}$, $(i\in I)$, there exists $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_h}\mathfrak{F}_\mathcal{A}$, $(h\in J)$, such that $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_i}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_h}\mathfrak{F}_\mathcal{A}$.

(ii) Additionally, $\sum_{j\in J}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A})\ \mathcal{R}\ \sum_{k\in K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{U}_k}\mathfrak{F}_\mathcal{A})$, so for every $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A}$, $(j\in J)$, there exists $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{U}_l}\mathfrak{F}_\mathcal{A}$, $(l\in K)$, such that $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{U}_l}\mathfrak{F}_\mathcal{A}$.

From (i) and (ii), we have, for every $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_i}\mathfrak{F}_\mathcal{A}$, $(i\in I)$, a $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{U}_l}\mathfrak{F}_\mathcal{A}$, $(l\in K)$, such that $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_i}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{U}_l}\mathfrak{F}_\mathcal{A}$. Simillarly, we have to prove that, for every $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{U}_k}\mathfrak{F}_\mathcal{A}$, $(k\in K)$, there exists $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_m}\mathfrak{F}_\mathcal{A}$, $(m\in I)$, such that $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{U}_k}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_m}\mathfrak{F}_\mathcal{A}$. This implies that $\sum_{i\in I}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_i}\mathfrak{F}_\mathcal{A})\ \mathcal{R}\ \sum_{k\in K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{U}_k}\mathfrak{F}_\mathcal{A})$. Hence, $\mathcal{R}$ is an equivalence relation on the ES structure. □

Proposition 1. *If $\sum_{i\in I}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_i}\mathfrak{F}_\mathcal{A}) \in ES$, $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_m}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_n}\mathfrak{F}_\mathcal{A}$, $n, m \in I$, $n \neq m$, then*

$$\textstyle\sum_{i\in I-\{m\}}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_i}\mathfrak{F}_\mathcal{A}) = \sum_{i\in I}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_i}\mathfrak{F}_\mathcal{A}).$$

Similarly, $\left[\sum_{i\in I-\{m\}}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_i}\mathfrak{F}_\mathcal{A})\right]_\mathcal{R} = \left[\sum_{i\in I}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_i}\mathfrak{F}_\mathcal{A})\right]_\mathcal{R}$, where $\left[\sum_{i\in I}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_i}\mathfrak{F}_\mathcal{A})\right]_\mathcal{R}$ is an equivalence class of $\sum_{i\in I}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_i}\mathfrak{F}_\mathcal{A}) \in ES$ (see Example 5).

Example 5. *Consider $\mathcal{S}_1 = \{\mathfrak{F}^1_\mathcal{A}, \mathfrak{F}^2_\mathcal{B}\}$, $\mathcal{S}_2 = \{\mathfrak{F}^1_\mathcal{A}, \mathfrak{F}^2_\mathcal{B}, \mathfrak{F}^3_\mathcal{C}\}$ and $\sum_{i\in\{1,2\}}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_i}\mathfrak{F}_\mathcal{A}) = \mathfrak{F}^1_\mathcal{A} \tilde{\wedge} \mathfrak{F}^2_\mathcal{B} + \mathfrak{F}^1_\mathcal{A} \tilde{\wedge} \mathfrak{F}^2_\mathcal{B} \tilde{\wedge} \mathfrak{F}^3_\mathcal{C}$. Let $\mathfrak{F}^1_\mathcal{A} \tilde{\wedge} \mathfrak{F}^2_\mathcal{B} = \mathfrak{J}_\mathcal{D}$, where $\mathcal{D} = \mathcal{A}\times\mathcal{B}$ and $\mathcal{J}(\alpha_1, \alpha_2) = \mathfrak{F}^1(\alpha_1) \cap \mathfrak{F}^2(\alpha_2)$, and $\mathfrak{F}^1_\mathcal{A} \tilde{\wedge} \mathfrak{F}^2_\mathcal{B} \tilde{\wedge} \mathfrak{F}^3_\mathcal{C} = \mathfrak{K}_\mathcal{M}$, where $\mathcal{M} = \mathcal{A}\times\mathcal{B}\times\mathcal{C}$ and $\mathcal{K}(\alpha_1, \alpha_2, \alpha_3) = \mathfrak{F}^1(\alpha_1) \cap \mathfrak{F}^2(\alpha_2) \cap \mathfrak{F}^3(\alpha_3)$.*

Now, We know that $\mathfrak{F}^1(\alpha_1) \cap \mathfrak{F}^2(\alpha_2) \cap \mathfrak{F}^3(\alpha_3) \subseteq \mathfrak{F}^1(\alpha_1) \cap \mathfrak{F}^2(\alpha_2)$ for all $\alpha_1 \in \mathcal{A}$, $\alpha_2 \in \mathcal{B}$, $\alpha_3 \in \mathcal{C}$. Therefore, the term $\mathfrak{F}^1_\mathcal{A} \tilde{\wedge} \mathfrak{F}^2_\mathcal{B} \tilde{\wedge} \mathfrak{F}^3_\mathcal{C}$ of given expression is redundant, and the expressions $\mathfrak{F}^1_\mathcal{A} \tilde{\wedge} \mathfrak{F}^2_\mathcal{B}$ and $\mathfrak{F}^1_\mathcal{A} \tilde{\wedge} \mathfrak{F}^2_\mathcal{B} \tilde{\wedge} \mathfrak{F}^3_\mathcal{C}$ are equivalent. Hence, $\sum_{i\in\{1,2\}}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_i}\mathfrak{F}_\mathcal{A}) = \mathfrak{F}^1_\mathcal{A} \tilde{\wedge} \mathfrak{F}^2_\mathcal{B}$.

In light of the interpretation of soft sets $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_k}\mathfrak{F}_\mathcal{A}$ and $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_j}\mathfrak{F}_\mathcal{A}$, we can see that the approximate value set for every attribute/parameter of soft set $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_j}\mathfrak{F}_\mathcal{A}$ is always a subset of or equal to that of $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_k}\mathfrak{F}_\mathcal{A}$ considering $\mathcal{T}_j \supseteq \mathcal{S}_k$. This implies that, if $\mathcal{T}_j \supseteq \mathcal{S}_k$ then $\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_j}\mathfrak{F}_\mathcal{A} \tilde{\subseteq}_J \prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_k}\mathfrak{F}_\mathcal{A}$.

Theorem 3. *Let $\tilde{\vee}_{ES}$, $\tilde{\wedge}_{ES}$ be two binary operations on ES. Then, (ES, $\tilde{\vee}_{ES}$, $\tilde{\wedge}_{ES}$) form a lattice, and $\tilde{\vee}_{ES}$, $\tilde{\wedge}_{ES}$ are defined as: for any $\sum_{j\in J}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A})$, $\sum_{k\in K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_k}\mathfrak{F}_\mathcal{A}) \in ES$,*

(i) $\left(\sum_{j\in J}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A})\right) \tilde{\wedge}_{ES} \left(\sum_{k\in K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_k}\mathfrak{F}_\mathcal{A})\right) = \sum_{j\in J,k\in K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j\cup\mathcal{T}_k}\mathfrak{F}_\mathcal{A})$,

(ii) $\left(\sum_{j\in J}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A})\right) \tilde{\vee}_{ES} \left(\sum_{k\in K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_k}\mathfrak{F}_\mathcal{A})\right) = \sum_{l\in J\cup K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{U}_l}\mathfrak{F}_\mathcal{A})$,
where $l \in J \cup K$ (disjoin union of J and K sets), $\mathcal{U}_l = \mathcal{S}_l$, if $l \in J$, and $\mathcal{U}_l = \mathcal{T}_l$, if $l \in K$.

Proof. In this proof, for $\sum_{j\in J}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A})$, $\sum_{k\in K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_k}\mathfrak{F}_\mathcal{A}) \in$ ES, $\sum_{j\in J}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A}) = \sum_{k\in K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_k}\mathfrak{F}_\mathcal{A})$ iff $\sum_{j\in J}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{S}_j}\mathfrak{F}_\mathcal{A})\ \mathcal{R}\ \sum_{k\in K}(\prod_{\mathfrak{F}_\mathcal{A}\in\mathcal{T}_k}\mathfrak{F}_\mathcal{A})$ (see Definition 6). The other part of the proof is similar to that of III-B-2 [17]. □

Result 2. *In Theorem 3, if sets $\mathcal{S}_j$ and $\mathcal{T}_k$ contain two or more soft sets, then the operators $\tilde{\vee}_{ES}$ and $\tilde{\wedge}_{ES}$ are different from $\tilde{\vee}$ and $\tilde{\wedge}$, respectively, of soft sets with respect to soft equal relation (Definition 4), and same with respect to the soft J-equal relation. When the sets J, K, $\mathcal{S}_j$ and $\mathcal{T}_k$ contain only one element, then the above operators are the same with respect to both soft equal and soft J-equal relations (see Example 6).*

Example 6. *Let $\mathcal{S}_1 = \{\mathfrak{F}^1_{\mathcal{A}}, \mathfrak{F}^2_{\mathcal{B}}\}$ and $\mathcal{T}_2 = \{\mathfrak{F}^2_{\mathcal{B}}, \mathfrak{F}^3_{\mathcal{C}}\}$. Then, from Theorem 3 (i), we have*

$$\textstyle\sum_{i\in\{1\}}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\ \tilde{\wedge}_{ES} \sum_{j\in\{2\}}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}}) = \sum_{i\in\{1\},j\in\{2\}}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i\cup\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}}).$$

Suppose that $\tilde{\wedge}_{ES}$ and $\tilde{\wedge}$ are the same operators. Then, $\sum_{i\in\{1\}}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\ \tilde{\wedge}_{ES} \sum_{j\in\{2\}}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}}) = (\mathfrak{F}^1_{\mathcal{A}}\ \tilde{\wedge}\ \mathfrak{F}^2_{\mathcal{B}})\ \tilde{\wedge}\ (\mathfrak{F}^2_{\mathcal{B}}\ \tilde{\wedge}\ \mathfrak{F}^3_{\mathcal{C}})$ and $\sum_{i\in\{1\},j\in\{2\}}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i\cup\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}}) = \mathfrak{F}^1_{\mathcal{A}}\ \tilde{\wedge}\ \mathfrak{F}^2_{\mathcal{B}}\ \tilde{\wedge}\ \mathfrak{F}^3_{\mathcal{C}}$ since from [12], $(\mathfrak{F}^1_{\mathcal{A}}\ \tilde{\wedge}\ \mathfrak{F}^2_{\mathcal{B}})\ \tilde{\wedge}\ (\mathfrak{F}^2_{\mathcal{B}}\ \tilde{\wedge}\ \mathfrak{F}^3_{\mathcal{C}}) \neq \mathfrak{F}^1_{\mathcal{A}}\ \tilde{\wedge}\ \mathfrak{F}^2_{\mathcal{B}}\ \tilde{\wedge}\ \mathfrak{F}^3_{\mathcal{C}}$ with respect to soft equal sets. However, with respect to soft J-equal sets, we have $(\mathfrak{F}^1_{\mathcal{A}}\ \tilde{\wedge}\ \mathfrak{F}^2_{\mathcal{B}})\ \tilde{\wedge}\ (\mathfrak{F}^2_{\mathcal{B}}\ \tilde{\wedge}\ \mathfrak{F}^3_{\mathcal{C}}) =_J \mathfrak{F}^1_{\mathcal{A}}\ \tilde{\wedge}\ \mathfrak{F}^2_{\mathcal{B}}\ \tilde{\wedge}\ \mathfrak{F}^3_{\mathcal{C}}$. Hence, $\tilde{\wedge}$ and $\tilde{\wedge}_{ES}$ are not the same operators.

Theorem 4. *If we define "$\leq_{ES}$" on ES as follows: for any $\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}}), \sum_{j\in J}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}}) \in ES$,*
$\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}}) \leq_{ES} \sum_{j\in J}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}}) \iff$

$$\left(\textstyle\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\right)\ \tilde{\vee}_{ES}\ \left(\sum_{j\in J}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}})\right) = \sum_{j\in J}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}})$$

if, and only if, for any $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}}$, $(i \in I)$, there exist $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_h}\mathfrak{F}_{\mathcal{A}}$, $(h \in J)$ such that $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}} \tilde{\subseteq}_J \prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_h}\mathfrak{F}_{\mathcal{A}}$, then "$\leq_{ES}$" is a partial relation on ES, and $(ES, \leq_{ES})$ is a distributive lattice.

Proof. Proof is similar to that of III-B-4 [17]. □

Here, we show some properties of lattice $(ES, \tilde{\vee}_{ES}, \tilde{\wedge}_{ES})$ from [17] with some modifications in note III-B-5.

1. $\left(\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\right)\ \tilde{\wedge}_{ES}\ \left(\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\right) = \sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})$;
2. $\left(\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\right)\ \tilde{\vee}_{ES}\ \left(\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\right) = \sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})$;
3. $\tilde{\phi}\ \tilde{\vee}_{ES}\ \left(\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\right) = \sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})$, where $\tilde{\phi}$ is a null soft set;
4. $\left(\sum_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}}\mathfrak{F}_{\mathcal{A}}\right)\ \tilde{\vee}_{ES}\ \left(\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\right) = \sum_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}}\mathfrak{F}_{\mathcal{A}}$;
5. $\tilde{\phi}\ \tilde{\wedge}_{ES}\ \left(\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\right) = \tilde{\phi}$;
6. $\left(\sum_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}}\mathfrak{F}_{\mathcal{A}}\right)\ \tilde{\wedge}_{ES}\ \left(\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\right) = \sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})$;
7. $\left[\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\ \tilde{\vee}_{ES} \sum_{j\in J}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}})\right]\ \tilde{\wedge}_{ES}\ \left[\sum_{k\in K}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{U}_k}\mathfrak{F}_{\mathcal{A}})\ \tilde{\vee}_{ES} \sum_{j\in J}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}})\right]$
$= \left[\sum_{i\in I}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}_i}\mathfrak{F}_{\mathcal{A}})\ \tilde{\wedge}_{ES} \sum_{k\in K}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{U}_k}\mathfrak{F}_{\mathcal{A}})\right]\ \tilde{\vee}_{ES} \sum_{j\in J}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{T}_j}\mathfrak{F}_{\mathcal{A}})$.

We can see that $\tilde{\phi}$ is the unit of $(ES, \tilde{\vee}_{ES})$ and $\sum_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}}\mathfrak{F}_{\mathcal{A}}$ is the unit of $(ES, \tilde{\wedge}_{ES})$. Additionally, $\tilde{\phi}$ is the minimal element and $\sum_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{S}}\mathfrak{F}_{\mathcal{A}}$ is the maximal element of the lattice $(ES, \tilde{\vee}_{ES}, \tilde{\wedge}_{ES})$. Moreover, if any $\sum_{k\in K}(\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{U}_k}\mathfrak{F}_{\mathcal{A}}) = \tilde{\phi} \in$ ES, then every $\prod_{\mathfrak{F}_{\mathcal{A}}\in\mathcal{U}_k}\mathfrak{F}_{\mathcal{A}} = \tilde{\phi}$, $k \in K$.

Algorithm 1

Provides the method for selecting suitable elements from universe set $\mathfrak{U}_s$ according to our requirement by using ES structure.

Input all soft sets, i.e. $\mathcal{S}$

$\Downarrow$

Input their corresponding parameter sets, i.e., $\mathcal{A} \subseteq \mathcal{E}$

$\Downarrow$

Construct all components of ES

$\Downarrow$

Select components of ES according to our requirements and place it in a tabular form

$\Downarrow$

Remove those parameters whose approximate sets are empty sets

$\Downarrow$

Choose the parameters according to the requirement

$\Downarrow$

Compute the corresponding approximate sets from the selected ES component and determine the result.

Examples 7 and 8 show how to solve a complex problem in decision-making.

Example 7. *Let $\mathfrak{U}_s = \{\mu_i \mid i \in \{1, 2, 3, 4, 5\}\}$ be the universe set, $\mathcal{P} = \{\wp_1^1, \wp_1^2, \wp_1^3, \wp_2^1, \wp_2^2, \wp_3^1, \wp_3^2, \wp_3^3, \wp_4^1, \wp_4^2, \wp_5^1, \wp_5^2\}$ be the set of all parameters and $\mathcal{S} = \{\mathfrak{F}_{\mathcal{A}}^1, \mathfrak{F}_{\mathcal{B}}^2, \mathfrak{F}_{\mathcal{C}}^3, \mathfrak{F}_{\mathcal{D}}^4, \mathfrak{F}_{\mathcal{E}}^5\}$ be a set of soft sets over $\mathfrak{U}_s$, given by the below Table 1:*

Table 1. Table of defined Soft sets over $\mathfrak{U}_s$.

$\mathcal{S}$	$\mathfrak{F}_{\mathcal{A}}^1$			$\mathfrak{F}_{\mathcal{B}}^2$		$\mathfrak{F}_{\mathcal{C}}^3$			$\mathfrak{F}_{\mathcal{D}}^4$		$\mathfrak{F}_{\mathcal{E}}^5$	
$\mathfrak{U}_s$ \ $\mathcal{P}$	$\wp_1^1$	$\wp_1^2$	$\wp_1^3$	$\wp_2^1$	$\wp_2^2$	$\wp_3^1$	$\wp_3^2$	$\wp_3^3$	$\wp_4^1$	$\wp_4^2$	$\wp_5^1$	$\wp_5^2$
μ_1	1	0	0	1	0	1	0	0	1	0	1	0
μ_2	0	1	0	1	0	0	1	0	1	0	1	1
μ_3	0	1	0	1	0	0	1	0	0	1	1	0
μ_4	0	0	1	0	1	0	0	1	0	1	1	1
μ_5	0	0	1	0	1	1	0	1	0	1	0	1

Let $\mathcal{S}_1 = \{\mathfrak{F}_{\mathcal{A}}^1, \mathfrak{F}_{\mathcal{B}}^2\} = \mathcal{T}_1$, $\mathcal{S}_2 = \{\mathfrak{F}_{\mathcal{B}}^2, \mathfrak{F}_{\mathcal{C}}^3\} = \mathcal{T}_2$ and $\mathcal{T}_3 = \{\mathfrak{F}_{\mathcal{A}}^1, \mathfrak{F}_{\mathcal{B}}^2, \mathfrak{F}_{\mathcal{C}}^3\}$. Then, $\prod_{\mathfrak{F}_{\mathcal{A}} \in \mathcal{S}_1} \mathfrak{F}_{\mathcal{A}} = \mathfrak{F}_{\mathcal{A}}^1 \tilde{\wedge} \mathfrak{F}_{\mathcal{B}}^2 = \mathfrak{F}_{\mathcal{A}\times\mathcal{B}}^{12}$, $\prod_{\mathfrak{F}_{\mathcal{A}} \in \mathcal{S}_2} \mathfrak{F}_{\mathcal{A}} = \mathfrak{F}_{\mathcal{B}}^2 \tilde{\wedge} \mathfrak{F}_{\mathcal{C}}^3 = \mathfrak{F}_{\mathcal{B}\times\mathcal{C}}^{23}$ and $\prod_{\mathfrak{F}_{\mathcal{A}} \in \mathcal{T}_3} \mathfrak{F}_{\mathcal{A}} = \mathfrak{F}_{\mathcal{A}}^1 \tilde{\wedge} \mathfrak{F}_{\mathcal{B}}^2 \tilde{\wedge} \mathfrak{F}_{\mathcal{C}}^3 = \mathfrak{F}_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}}^{123}$ are soft sets, where every touple $(\wp_k^i, \wp_l^j) \in \mathcal{A} \times \mathcal{B}$ can be written as $\wp_{kl}^{ij}$, $(i \in \{1, 2, 3\}, j, k, l \in \{1, 2\})$. Therefore, $\mathcal{A} \times \mathcal{B} = \{\wp_{12}^{11}, \wp_{12}^{12}, \wp_{12}^{21}, \wp_{12}^{22}, \wp_{12}^{31}, \wp_{12}^{32}\}$, $\mathcal{B} \times \mathcal{C} = \{\wp_{23}^{11}, \wp_{23}^{12}, \wp_{23}^{13}, \wp_{23}^{21}, \wp_{23}^{22}, \wp_{23}^{23}\}$ and $\mathcal{A} \times \mathcal{B} \times \mathcal{C} = \{\wp_{123}^{111}, \wp_{123}^{121}, \wp_{123}^{211}, \wp_{123}^{221}, \wp_{123}^{311}, \wp_{123}^{321}, \wp_{123}^{112}, \wp_{123}^{122}, \wp_{123}^{212}, \wp_{123}^{222}, \wp_{123}^{312}, \wp_{123}^{322}, \wp_{123}^{113}, \wp_{123}^{123}, \wp_{123}^{213}, \wp_{123}^{223}, \wp_{123}^{313}, \wp_{123}^{323}\}$. After utilizing Table 1 and removing those parameters whose approximate sets are empty sets, we obtained the following soft sets.

$$\mathfrak{F}_{\mathcal{A}\times\mathcal{B}}^{12} = \{(\wp_{12}^{11}, \{\mu_1\}), (\wp_{12}^{21}, \{\mu_2, \mu_3\}), (\wp_{12}^{32}, \{\mu_4, \mu_5\})\},$$

$$\mathfrak{F}_{\mathcal{B}\times\mathcal{C}}^{23} = \{(\wp_{23}^{11}, \{\mu_1\}), (\wp_{23}^{12}, \{\mu_2, \mu_3\}), (\wp_{23}^{21}, \{\mu_5\}), (\wp_{23}^{23}, \{\mu_4, \mu_5\})\},$$

$$\mathfrak{F}_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}}^{123} = \{(\wp_{123}^{111}, \{\mu_1\}), (\wp_{123}^{321}, \{\mu_5\}), (\wp_{123}^{212}, \{\mu_2, \mu_3\}), (\wp_{123}^{323}, \{\mu_4, \mu_5\})\}.$$

Take $\alpha = \sum_{i \in I}(\prod_{\mathfrak{F}_{\mathcal{A}} \in \mathcal{S}_i} \mathfrak{F}_{\mathcal{A}})$, $\gamma = \sum_{j \in J}(\prod_{\mathfrak{F}_{\mathcal{A}} \in \mathcal{T}_j} \mathfrak{F}_{\mathcal{A}}) \in ES$, $(i \in I = \{1, 2\}, j \in J = \{1, 2, 3\})$. Now, first we will show that α and γ are equivalent elements of ES. From the above soft sets, we can see that $\mathfrak{F}^{23}_{\mathcal{B}\times\mathcal{C}}$ and $\mathfrak{F}^{123}_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}}$ are soft J-equals, i.e., $\mathfrak{F}^{23}_{\mathcal{B}\times\mathcal{C}} =_J \mathfrak{F}^{123}_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}}$. Hence, using Proposition 1, α and γ are equivalent elements. We chose the elements from universe set $\mathfrak{U}_s$ using the choice parameters (($\wp^1_1$ and $\wp^1_2$) or ($\wp^1_2$ and $\wp^2_3$) or ($\wp^3_1$ and $\wp^2_2$ and $\wp^1_3$)). We solved this as:

$$\mathfrak{F}^{12}_{\mathcal{A}\times\mathcal{B}}(\wp^{11}_{12}) \cup \mathfrak{F}^{23}_{\mathcal{B}\times\mathcal{C}}(\wp^{12}_{23}) \cup \mathfrak{F}^{123}_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}}(\wp^{321}_{123}) = \{\mu_1, \mu_2, \mu_3, \mu_5\}.$$

Hence, μ_1, μ_2, μ_3 and μ_5 are four suitable elements of the universe set.

Example 8. *If a company has five machines, we consider it a universe set, defined as $\mathfrak{U}_s = \{m_i \mid i \in \{1, 2, 3, 4, 5\}\}$, and $\mathcal{P} = \{\wp^1_1, \wp^2_1, \wp^3_1, \wp^1_2, \wp^2_2, \wp^1_3, \wp^2_3, \wp^3_3\}$ are the set of parameters defined on $\mathfrak{U}_s$, where $\wp^i_1$, $\wp^i_2$ and $\wp^i_3$ represent shapes, stability and prizes repectively. "Mr. Q" comes to the company and wants to buy machines according to his requirements. He wants machines with either "$\wp^2_1$ shape and $\wp^2_2$ stability" or "$\wp^1_2$ stability and $\wp^2_3$ prize"; we write his requirement as, (($\wp^2_1$, $\wp^2_2$) + ($\wp^1_2$, $\wp^2_3$)). Therefore, we can see that he has complex requirements when selecting suitable machines. To solve this, we first consider a set of three soft sets, $\mathcal{S} = \{\mathfrak{F}^1_{\mathcal{A}}, \mathfrak{F}^2_{\mathcal{B}}, \mathfrak{F}^3_{\mathcal{C}}\}$ over $\mathfrak{U}_s$, defined in the below Table 2 as:*

Table 2. Table of Soft Sets in $\mathcal{S}$ over $\mathfrak{U}_s$.

$\mathcal{S}$	$\mathfrak{F}^1_{\mathcal{A}}$			$\mathfrak{F}^2_{\mathcal{B}}$			$\mathfrak{F}^3_{\mathcal{C}}$	
$\mathcal{P}$ / $\mathfrak{U}_s$	$\wp^1_1$	$\wp^2_1$	$\wp^3_1$	$\wp^1_2$	$\wp^2_2$	$\wp^1_3$	$\wp^2_3$	$\wp^3_3$
m_1	1	0	0	1	0	1	0	0
m_2	0	1	0	1	1	0	1	0
m_3	0	1	0	1	0	0	1	0
m_4	0	0	1	0	1	0	0	1
m_5	0	0	1	0	1	1	0	1

Now, $ES = \{\sum_{i \in I}(\prod_{\mathfrak{F}_{\mathcal{A}} \in \mathcal{S}_i} \mathfrak{F}_{\mathcal{A}}) \mid \mathcal{S}_i \subseteq \mathcal{S}\}$. For the requirements, of Mr. Q, i.e., (($\wp^2_1$, $\wp^2_2$) + ($\wp^1_2$, $\wp^2_3$)), we chose $\mathcal{S}_1 = \{\mathfrak{F}^1_{\mathcal{A}}, \mathfrak{F}^2_{\mathcal{B}}\}$ and $\mathcal{S}_2 = \{\mathfrak{F}^2_{\mathcal{B}}, \mathfrak{F}^3_{\mathcal{C}}\}$. Further, an element can be selected $\sum_{i \in I}(\prod_{\mathfrak{F}_{\mathcal{A}} \in \mathcal{S}_i} \mathfrak{F}_{\mathcal{A}}) \in ES$, $(i \in \{1, 2\})$, such as:

$$\prod_{\mathfrak{F}_{\mathcal{A}} \in \mathcal{S}_1} \mathfrak{F}_{\mathcal{A}} = \mathfrak{F}^{12}_{\mathcal{A}\mathcal{B}} = \mathfrak{F}^1_{\mathcal{A}} \tilde{\wedge} \mathfrak{F}^2_{\mathcal{B}}, \text{ and}$$
$$\prod_{\mathfrak{F}_{\mathcal{A}} \in \mathcal{S}_2} \mathfrak{F}_{\mathcal{A}} = \mathfrak{F}^{23}_{\mathcal{B}\mathcal{C}} = \mathfrak{F}^2_{\mathcal{B}} \tilde{\wedge} \mathfrak{F}^3_{\mathcal{C}}$$

The requirement is placed in the ES selected element and using Table 2, we have:

$$\begin{aligned} &\sum_{i \in I}(\prod_{\mathfrak{F}_{\mathcal{A}} \in \mathcal{S}_i} \mathfrak{F}_{\mathcal{A}})((\wp^2_1, \wp^2_2) + (\wp^1_2, \wp^2_3)) \\ &= \mathfrak{F}^{12}_{\mathcal{A}\mathcal{B}}(\wp^2_1, \wp^2_2) \cup \mathfrak{F}^{23}_{\mathcal{B}\mathcal{C}}(\wp^1_2, \wp^2_3) \\ &= (\mathfrak{F}^1_{\mathcal{A}}(\wp^2_1) \cap \mathfrak{F}^2_{\mathcal{B}}(\wp^2_2)) \cup (\mathfrak{F}^2_{\mathcal{B}}(\wp^1_2) \cap \mathfrak{F}^3_{\mathcal{C}}(\wp^2_3)) \\ &= \{m_2, m_3\}. \end{aligned}$$

Hence, Mr. Q will select two machines, m_2 and m_3, according to his requirements.

4. Result and Concluding Remarks

In the existing literature [17], the construction of a structure ES using all soft sets, $\mathfrak{F}_{\mathcal{A}} \in \mathcal{S}$, is defined over a common universe $\mathfrak{U}_s$ and forms a complete lattice by utilizing the soft M-subset. However, due to parameter set restrictions in soft M-subsets, it cannot provide a complete lattice leading to a roadblock in this regard. To obtain the desired result and required solution, we selected the soft J-subset. The soft J-subset provides us with results, due to the relaxation of restrictions on the parameter set. Hence, our study has better applicability, as it is easy to use when defining binary operations on ES. Using the same notion, we designed an algorithm to understand the concept, and a real-life example is provided using the methodology. The findings of this study lead us to the conclusion that a significant number of complicated soft sets, i.e., the elements of ES structure, can be described by some soft sets in $\mathcal{S}$, and that their logic operations will be useful when solving complex problems in decision-making. In the ES structure, we start by considering all the potential conjunctions ($\tilde{\wedge}$) and disjunctions ($\tilde{\vee}$) of soft sets defined on a shared universe $\mathfrak{U}_s$, and we search the universal set for items that satisfy the established criteria. One conclusion that can be drawn from this is that the research sheds light on the cognitive field in a more precise manner. At the very end, we presented an illustration using an example showing how to solve a complex problem involving decision-making in an ES structure. In the future, one of our goals is to investigate the structure's order-reversing involution. Next, our efforts will consist of constructing operations that will allow for us to proceed toward a boolean algebra.

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