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On the Joint A-Numerical Radius of Operators and Related Inequalities

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Abstract: In this paper, we study p-tuples of bounded linear operators on a complex Hilbert space with adjoint operators defined with respect to a non-zero positive operator A. Our main objective is to investigate the joint A-numerical radius of the p-tuple.We established several upper bounds for it, some of which extend and improve upon a previous work of the second author. Additionally, we provide several sharp inequalities involving the classical A-numerical radius and the A-seminorm of semi-Hilbert space operators as applications of our results.

Keywords: positive operator; joint A-numerical radius; Euclidean operator A-seminorm; joint operator A-seminorm

MSC: 47B65; 47A12; 47A13; 47A30

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1. Introduction

In recent years, there has been growing interest in the study of operators on semi-Hilbert spaces, as evidenced by works such as [1–6] and related literature. This area of research is quite promising as a subfield of functional analysis. One reason for the renewed interest in the semi-Hilbert analysis is that it provides a more general framework for defining operators that represent physical observables in quantum mechanics (QM). In standard QM, the physical states of a quantum system are represented on a Hilbert space $\mathscr{H}$ with a given inner product $\langle\cdot,\cdot\rangle$. Typically, operators representing physical observables should be self-adjoint with respect to the given inner product, which is somewhat restrictive. However, the theory of non-Hermitian QM offers a more general approach that defines a new inner product using a metric operator A, such that $\langle\xi,\eta\rangle_A = \langle A\xi,\eta\rangle$ for any $\xi,\eta \in \mathscr{H}$, and the considered operators are self-adjoint with respect to this new inner product. In quasi-Hermitian QM [7,8] and pseudo-Hermitian QM [9], the metric operator A is invertible, self-adjoint, and positive, with respect to the reference inner product. In contrast, in indefinite metric QM [10], the underlying operator is unitary and self-adjoint but not necessarily positive. In the mathematical approach, the operator A is self-adjoint and positive with respect to the usual inner product of $\mathscr{H}$, but it is not necessarily invertible.

Motivated by the study of operators in the context of quantum mechanics, researchers have recently been very interested in the joint A-numerical radius and related inequalities. This concept extends the joint numerical radius of operators in Hilbert spaces. Specifically, when $A = I$, we obtain the definition of the joint numerical radius of operators in Hilbert spaces.

There are many other problems worth exploring in numerical ranges and radii for both single and multivariable operators in Hilbert spaces. These include investigating topics, such as operator convergence properties, functional equations, operator trigonometry, model theory, robust stability, reduction theory, and factorization of matrix polynomials. Additionally, intrinsic problems, such as the convexity of various types of generalized numerical ranges, the realizability of certain sets (such as the numerical ranges of an operator), the completability of partial matrices, and the classification of linear preservers are of interest. For more information on some of these applications, interested readers may refer to the following references, such as [11,12], and the references within. The applications mentioned above have motivated us to explore the connection between the A-joint numerical radius of operators and other areas of applied mathematics. This highlights the significance of studying the A-joint numerical radius of operators.

Another crucial motivation for our current study involves recent research that has focused on developing numerical radius inequalities for both single and multivariable Hilbert space operators, including the joint numerical range and numerical radius. Developing such inequalities has broad implications for applications in functional analysis and the operator theory (see, for example, [2], which contains a wealth of additional resources on this topic). In particular, the study of the A-joint numerical radius of operators in Hilbert spaces is a relatively new and important area of research that has gained increasing interest among researchers in recent years. Mathematical inequalities involving the A-joint numerical radius are essential tools for understanding the behaviors of these operators and their applications, as seen in recent research (e.g., see [13] and its extensive reference list).

In this paper, our focus is on studying the joint A-numerical radius of bounded linear operators on a complex Hilbert space, which is a generalization of the numerical radius of operators in Hilbert spaces. This quantity is defined with respect to a non-zero positive operator A. Our main objective is to establish upper bounds for the joint A-numerical radius and provide several sharp inequalities that involve the classical A-numerical radius and the A-seminorm of semi-Hilbert space operators. By doing so, we aim to contribute to the existing body of knowledge in the field of functional analysis and operator theory.

2. Notations and Preliminary Results

In this section, we introduce the notations and preliminary results that will be used throughout the article. To begin with, we denote by $\mathscr{L}(\mathscr{H})$ the Banach algebra of all bounded linear operators acting on a complex Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ with the identity operator $I_{\mathscr{H}}$. The norm induced by $\langle \cdot, \cdot \rangle$ is given by $\|\xi\| = \sqrt{\langle \xi, \xi \rangle}$ for all $\xi \in \mathscr{H}$. The range, the null space, and the adjoint of an operator $X \in \mathscr{L}(\mathscr{H})$ are, respectively, denoted by $\mathcal{R}(X)$, $\mathcal{N}(X)$, and X^*. By $\overline{\mathcal{R}(X)}$, we mean the norm closure of the subspace $\mathcal{R}(X)$. Further, we recall that the cone of every positive operator is defined as:

$$\mathscr{L}^+(\mathscr{H}) = \left\{X \in \mathscr{L}(\mathscr{H})\,;\, \langle X\xi, \xi \rangle \geq 0, \quad \forall\, \xi \in \mathscr{H}\right\}.$$

If $X \in \mathscr{L}^+(\mathscr{H})$, then we write $X \geq 0$. By $X^{\frac{1}{2}}$, we mean the square root of every $X \in \mathscr{L}^+(\mathscr{H})$. For the rest of the present paper, we retain the notation A for a non-zero operator in $\mathscr{L}^+(\mathscr{H})$, which defines the following positive (semidefinite) sesquilinear form:

$$\langle \cdot, \cdot \rangle_A : \mathscr{H} \times \mathscr{H} \longrightarrow \mathbb{C}, (\xi, \eta) \mapsto \langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle = \langle A^{\frac{1}{2}}\xi, A^{\frac{1}{2}}\eta \rangle.$$

The seminorm induced by $\langle \cdot, \cdot \rangle_A$ is defined as $\|\xi\|_A = \sqrt{\langle \xi, \xi \rangle_A}$ for all $\xi \in \mathscr{H}$. Let S_1^A stand for the A-unit sphere of $\mathscr{H}$, i.e.,

$$S_1^A = \{\xi \in \mathscr{H}\,;\, \|\xi\|_A = 1\}.$$

Note that $(\mathscr{H}, \|\cdot\|_A)$ is called a semi-Hilbert space, which is generally neither a normed space nor a complete space (see [14]).

We use the notation $\mathbb{N}^*$ to represent the set of all positive integers. Let p be an element of $\mathbb{N}^*$. In accordance with [3], we introduce the joint A-numerical range and joint A-numerical radius associated with the p-tuples of operators $\mathbf{X} = \left(X_1, \ldots, X_p\right) \in \mathscr{L}(\mathscr{H})^{(p)}$, where $\mathscr{L}(\mathscr{H})^{(p)}$ denotes the direct sum of p copies of the operator space $\mathscr{L}(\mathscr{H})$. The joint A-numerical range, denoted by $JtW_A(\mathbf{X})$, is defined as

$$JtW_A(\mathbf{X}) := \left\{ \left(\langle X_1\xi, \xi\rangle_A, \ldots, \langle X_p\xi, \xi\rangle_A \right) ;\ \xi \in S_1^A \right\}.$$

Similarly, the joint A-numerical radius, denoted by $\omega_{\mathrm{e},A}(\mathbf{X})$, is defined as

$$\begin{aligned} \omega_{\mathrm{e},A}(\mathbf{X}) &= \sup\left\{ \|\lambda\|_2 := \left(\sum_{j=1}^{p} |\lambda_j|^2 \right)^{\frac{1}{2}} ;\ \lambda = (\lambda_1, \ldots, \lambda_p) \in JtW_A(\mathbf{X}) \right\} \\ &= \sup_{\xi \in S_1^A} \left(\sum_{m=1}^{p} |\langle X_m\xi, \xi\rangle_A|^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{1}$$

It is crucial to mention that $\omega_A(\mathbf{X})$ may be equal to $+\infty$ for certain p-tuples of operators $\mathbf{X} \in \mathscr{L}(\mathscr{H})^{(p)}$ even if $p = 1$ (for instance, see [14]). Several interesting properties involving the joint A-numerical radius $\omega_A(\cdot)$ of A-bounded operators were stated in [3,15]. A recent investigation of $\omega_A(\cdot)$ for $d = 2$ was provided by the third author in [16]. By setting $d = 1$ in (1), we obtain the well-known A-numerical radius of an operator $X \in \mathscr{L}(\mathscr{H})$, which was firstly defined in [17]. Namely, we have

$$\omega_A(X) = \sup_{\xi \in S_1^A} \left| \langle X_m\xi, \xi\rangle_A \right|.$$

Many fundamental characteristics of the A-numerical radius of operators can be discovered in various sources, such as [2,3,6,18,19], and the related literature.

Recall from [20] that an operator $Y \in \mathscr{L}(\mathscr{H})$ is called an A-adjoint of an operator $X \in \mathscr{L}(\mathscr{H})$ if $\langle X\xi, \eta\rangle_A = \langle \xi, Y\eta\rangle_A$ for every $\xi, \eta \in \mathscr{H}$. In other words, Y is a solution in $\mathscr{L}(\mathscr{H})$ of the equation $AZ = X^*A$. Notice that an operator $X \in \mathscr{L}(\mathscr{H})$ does not generally admit an A-adjoint, and even if X has an A-adjoint Y, then Y is not generally unique. By $\mathscr{L}_A(\mathscr{H})$, we denote the set of all bounded linear operators on $\mathscr{H}$ that admit A-adjoints. The well-known Douglas theorem [21] assures the existence of such sets of operators. More precisely, by the Douglas theorem [21], we have

$$\mathscr{L}_A(\mathscr{H}) = \{X \in \mathscr{L}(\mathscr{H});\ \mathcal{R}(X^*A) \subseteq \mathcal{R}(A)\}.$$

In addition, another application of the Douglas theorem [21] shows that if $X \in \mathscr{L}_A(\mathscr{H})$, then the equation $AZ = X^*A$ has a unique solution in $\mathscr{L}(\mathscr{H})$, denoted by $X^{\sharp_A}$, satisfying $\mathcal{R}\left(X^{\sharp_A}\right) \subseteq \overline{\mathcal{R}(A)}$. The operator $X^{\sharp_A}$ may be computed via the following formula: $X^{\sharp_A} = A^{\dagger}X^*A$, where $A^{\dagger}$ denotes the Moore–Penrose inverse of A (see [20]). The operator $X^{\sharp_A}$ has similar but not identical properties to $X^* := X^{\sharp_I}$. In particular, if $X \in \mathscr{L}_A(\mathscr{H})$, then so does $X^{\sharp_A}$. Furthermore, we have

$$(X^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}} X P_{\overline{\mathcal{R}(A)}} \quad \text{and} \quad \left((X^{\sharp_A})^{\sharp_A}\right)^{\sharp_A} = X^{\sharp_A}. \tag{2}$$

Moreover, in view of [22], the following equalities

$$\omega_A(X^{\sharp_A}) = \omega_A(X) = \omega_A\left(P_{\overline{\mathcal{R}(A)}}X\right) = \omega_A\left(XP_{\overline{\mathcal{R}(A)}}\right), \tag{3}$$

hold for every $X \in \mathscr{L}_A(\mathscr{H})$. In addition, we mention that for $X, Y \in \mathscr{L}_A(\mathscr{H})$, we have $XY \in \mathscr{L}_A(\mathscr{H})$ and $(XY)^{\sharp_A} = Y^{\sharp_A}X^{\sharp_A}$. Now, let $X \in \mathscr{L}_A(\mathscr{H})$. The operator X is said to be

A-self-adjoint if AX is self-adjoint, i.e., $AX = X^*A$. Note that the class of A-self-adjoint operators does not cover the equality between X and $X^{\sharp_A}$. However, according to [20], we have $X = X^{\sharp_A}$ if and only if X is an A-self-adjoint operator and $\mathcal{R}(X) \subseteq \overline{\mathcal{R}(A)}$. Now, we should note that X is A-positive and we simply write $X \geq_A 0$ if $AX \in \mathscr{L}(\mathscr{H})^+$. Clearly, if an operator X is A-self-adjoint, then $X \in \mathscr{L}_A(\mathscr{H})$. It is proved in [23] that if $X \in \mathscr{L}(\mathscr{H})$ is A-self-adjoint, then so is $X^{\sharp_A}$, and the following property

$$(X^{\sharp_A})^{\sharp_A} = X^{\sharp_A}, \tag{4}$$

holds. An operator $X \in \mathscr{L}_A(\mathscr{H})$ is referred to as an A-normal operator if and only if $XX^{\sharp_A} = X^{\sharp_A}X$. While it is well-known that all self-adjoint operators in a Hilbert space are normal, this fact may not hold true for A-self-adjoint operators. In other words, A-self-adjoint operators may not necessarily be A-normal, as shown in [3] (Example 5.1) or [14].

In the present work, we denote by

$$\Re_A(Q) := \frac{Q + Q^{\sharp_A}}{2} \quad \text{and} \quad \Im_A(Q) := \frac{Q - Q^{\sharp_A}}{2i},$$

the A-real and A-imaginary parts of an operator $Q \in \mathscr{L}_A(\mathscr{H})$, respectively. It is clear that for every $X \in \mathscr{L}_A(\mathscr{H})$, we have $X = \Re_A(X) + i\Im_A(X)$.

If $A \geq 0$, then obviously $A^{\frac{1}{2}} \geq 0$. Let $\mathscr{L}_{A^{\frac{1}{2}}}(\mathscr{H})$ stand for the set of all operators in $\mathscr{H}$ that admit $A^{\frac{1}{2}}$-adjoints. Again, the Douglas theorem [21] guarantees that

$$\mathscr{L}_{A^{\frac{1}{2}}}(\mathscr{H}) = \left\{X \in \mathscr{L}(\mathscr{H})\,;\, \|X\xi\|_A \leq \lambda\|\xi\|_A, \text{ for some } \lambda > 0 \text{ and all } \xi \in \mathscr{H}\right\}.$$

Operators in $\mathscr{L}_{A^{\frac{1}{2}}}(\mathscr{H})$ are called A-bounded. We should note that the following inclusions

$$\mathscr{L}_A(\mathscr{H}) \subseteq \mathscr{L}_{A^{\frac{1}{2}}}(\mathscr{H}) \subseteq \mathscr{L}(\mathscr{H})$$

hold. We should note that the above inclusions are generally strict. However, the equality between the above sets holds if A is injective and has a closed range in $\mathscr{H}$. Notice that $\mathscr{L}_A(\mathscr{H})$ and $\mathscr{L}_{A^{\frac{1}{2}}}(\mathscr{H})$ are two subalgebras of $\mathscr{L}(\mathscr{H})$. However, they are generally not closed and not dense in $\mathscr{L}(\mathscr{H})$ (see [20]).

If $X \in \mathscr{L}_{A^{\frac{1}{2}}}(\mathscr{H})$, then the A-seminorm of X is given by:

$$\|X\|_A = \sup_{\substack{\xi \in \overline{\mathcal{R}(A)} \\ \xi \neq 0}} \frac{\|X\xi\|_A}{\|\xi\|_A} = \sup_{\xi \in S_1^A} \|X\xi\|_A = \sup_{\xi,\eta \in S_1^A} \left|\langle X\xi, \eta\rangle_A\right|. \tag{5}$$

If $X \in \mathscr{L}(\mathscr{H}) \backslash \mathscr{L}_{A^{\frac{1}{2}}}(\mathscr{H})$, then it may happen that $\|X\|_A = +\infty$ (see [14]). It follows from (5) that the equality $\|X\|_A = \left\|X^{\sharp_A}\right\|_A$ holds for every $X \in \mathscr{L}_A(\mathscr{H})$. If X is an A-self-adjoint operator (in particular if $X \geq_A 0$), then

$$\omega_A(X) = \|X\|_A, \tag{6}$$

and

$$\|X^n\|_A = \|X\|_A^n, \tag{7}$$

for every $n \in \mathbb{N}^*$. It is useful to note that for every $X \in \mathscr{L}_A(\mathscr{H})$, we have $X^{\sharp_A}X \geq_A 0$ and $XX^{\sharp_A} \geq_A 0$. Therefore, we can obtain the following result by applying (6) in conjunction with the last equality in (5):

$$\left\|X^{\sharp_A}X\right\|_A = \left\|XX^{\sharp_A}\right\|_A = \|X\|_A^2 = \left\|X^{\sharp_A}\right\|_A^2. \tag{8}$$

Baklouti et al. introduced in [3] an extension of (5) that applies to tuples of A-bounded operators. Specifically, they defined the joint A-seminorm of the p-tuples of operators $\mathbf{X} = (X_1, \dots, X_p) \in \mathscr{L}^{(p)}_{A^{\frac{1}{2}}}$ as

$$\|\mathbf{X}\|_A = \sup_{\xi \in S_1^A} \left(\sum_{m=1}^{p} \|X_m \xi\|_A^2 \right)^{\frac{1}{2}}. \tag{9}$$

If $X_m \in \mathscr{L}_A(\mathscr{H})$ for all $m \in \{1, \dots, p\}$, then we remark that $\sum_{k=1}^{p} X_k^{\sharp_A} X_k \geq_A 0$. Consequently, by using (6), we can deduce that

$$\|\mathbf{X}\|_A = \left\| \sum_{k=1}^{d} X_k^{\sharp_A} X_k \right\|_A^{\frac{1}{2}}.$$

It is convenient to note that $\|\cdot\|_A$ and $\omega_{e,A}(\cdot)$ defines two equivalent seminorms on $\mathscr{L}^{(p)}_{A^{\frac{1}{2}}}$. More precisely, for $\mathbf{X} = (X_1, \dots, X_p) \in \mathscr{L}_{A^{\frac{1}{2}}}(\mathscr{H})^p$, it was shown in [3] that

$$\frac{1}{2\sqrt{p}} \|\mathbf{X}\|_A \leq \omega_{e,A}(\mathbf{X}) \leq \|\mathbf{X}\|_A. \tag{10}$$

In particular, if $\mathbf{X} = (X_1, \dots, X_p) \in \mathscr{L}_A(\mathscr{H})^p$, then we have

$$\frac{1}{4p} \left\| \sum_{k=1}^{p} X_k^{\sharp_A} X_k \right\|_A \leq \omega_{e,A}^2(\mathbf{X}) \leq \left\| \sum_{k=1}^{p} X_k^{\sharp_A} X_k \right\|_A. \tag{11}$$

Building upon the recent research of the third author in [16] and the work of the second author in [24], this article establishes several new inequalities for the joint A-numerical radius of semi-Hilbert space operators. To achieve this, we utilize extensions of the well-known Bessel inequality developed by Bombieri, the third author, and Boas–Bellman.

The implications of our results extend beyond the specific context of semi-Hilbert space operators. As a particular application, we present sharp bounds for the classical A-numerical radius. These findings contribute to the ongoing research in operator theory and functional analysis, and we expect that they will inspire further exploration of this topic.

3. Main Results

In this section, we will present the main findings of our study. We will start by introducing a key lemma that plays a crucial role in the proof of our first result.

Lemma 1. *Let $y_1, \dots, y_p$ be vectors in $\mathscr{H}$. Then, for all $x \in \mathscr{H}$, we have*

$$\sum_{i=1}^{p} \left| \langle x, y_i \rangle_A \right|^2 \leq \|x\|_A^2 \left(\sum_{i,j=1}^{p} \left| \langle y_i, y_j \rangle_A \right|^2 \right)^{\frac{1}{2}}.$$

Proof. Recall the following inequality from [25]:

$$\sum_{i=1}^{p}\left|\langle a,b_i\rangle\right|^2 \le \|x\|^2\left(\sum_{i,j=1}^{p}\left|\langle b_i,b_j\rangle\right|^2\right)^{\frac{1}{2}}, \tag{12}$$

which holds for any $a, b_1, \dots, b_p \in \mathscr{H}$. Now, let $x, y_1, \dots, y_p$ be vectors in $\mathscr{H}$. By letting $a = A^{\frac{1}{2}}$ and $b_k = A^{\frac{1}{2}}y_k$ for all $k \in \{1, \dots, p\}$ in (12), we see that

$$\begin{aligned}\sum_{i=1}^{p}\left|\langle x,y_i\rangle_A\right|^2 &= \sum_{i=1}^{p}\left|\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}y_i\rangle\right|^2 \\ &\le \left\|A^{\frac{1}{2}}x\right\|^2\left(\sum_{i,j=1}^{p}\left|\langle A^{\frac{1}{2}}y_i, A^{\frac{1}{2}}y_j\rangle\right|^2\right)^{\frac{1}{2}} \\ &= \|x\|_A^2\left(\sum_{i,j=1}^{p}\left|\langle y_i,y_j\rangle_A\right|^2\right)^{\frac{1}{2}}.\end{aligned}$$

So, we obtain the desired result. □

We are pleased to introduce our first result, which gives an upper bound for the joint A-numerical radius of operators. The result is stated as follows:

Theorem 1. *Let* $\mathbf{X} = (X_1, \dots, X_p) \in \mathscr{L}_A(\mathscr{H})^{(p)}$. *Then*

$$\begin{aligned}\omega_{e,A}^2(\mathbf{X}) &\le \left[\omega_{e,A}^2(\mathbf{Y}) + \sum_{1\le i\ne j\le n}\omega_A^2(X_j^{\sharp_A}X_i)\right]^{\frac{1}{2}} \\ &\le \left[\left\|\sum_{i=1}^{p}\left(X_i^{\sharp_A}X_i\right)^2\right\|_A + \sum_{1\le i\ne j\le p}\omega_A^2\left(X_j^{\sharp_A}X_i\right)\right]^{\frac{1}{2}},\end{aligned}$$

where $\mathbf{Y} = (X_1^{\sharp_A}X_1, \dots, X_p^{\sharp_A}X_p)$.

Proof. Let $\xi \in S_1^A$. By applying Lemma 1, for $x = \xi$, and $y_m = X_m\xi$ for all $m \in \{1, \dots, p\}$, we see that

$$\begin{aligned}\sum_{i=1}^{p}\left|\langle X_i\xi,\xi\rangle_A\right|^2 &\le \left[\sum_{i=1}^{p}\|X_i\xi\|_A^4 + \sum_{1\le i\ne j\le p}\left|\langle X_i\xi, X_j\xi\rangle_A\right|^2\right]^{\frac{1}{2}} \\ &= \left[\sum_{i=1}^{p}\left|\langle X_i^{\sharp_A}X_i\xi,\xi\rangle_A\right|^2 + \sum_{1\le i\ne j\le p}\left|\langle X_i\xi, X_j\xi\rangle_A\right|^2\right]^{\frac{1}{2}} \\ &= \left[\sum_{i=1}^{p}\left|\langle X_i^{\sharp_A}X_i\xi,\xi\rangle_A\right|^2 + \sum_{1\le i\ne j\le p}\left|\langle X_j^{\sharp_A}X_i\xi,\xi\rangle_A\right|^2\right]^{\frac{1}{2}}.\end{aligned}$$

Let $\mathbf{Y} = (X_1^{\sharp_A}X_1, \dots, X_p^{\sharp_A}X_p)$. One observes that

$$\sum_{i=1}^{p}\left|\langle X_i\xi,\xi\rangle_A\right|^2 \le \left[\omega_{\mathrm{e},A}^2(\mathbf{Y}) + \sum_{1\le i\ne j\le n}\omega_A^2(X_j^{\sharp_A}X_i)\right]^{\frac{1}{2}}.$$

By taking the supremum over all $\xi \in S_1^A$ in the above inequality, we reach the first inequality in Theorem 1. On the other hand, it is clear that $Y_k \geq_A 0$ for all $k \in \{1, \ldots, p\}$. This yields that Y_k is an A-self-adjoint operator for all k. Further, since $\mathcal{R}(Y_k) \subseteq \overline{\mathcal{R}(A)}$ for all k, then

$$Y_k^{\sharp_A} = (X_k^{\sharp_A} X_k)^{\sharp_A} = X_k^{\sharp_A} X_k, \quad \forall k \in \{1, \ldots, p\}.$$

Therefore, we can conclude that the second inequality in Theorem 1 is a direct consequence of the second inequality in (11). Hence, the proof is complete. □

Based on the above result, we can derive several corollaries. The first corollary is presented below.

Corollary 1. *Let* $\mathbf{X} = \big(X_1, \ldots, X_p\big) \in \mathscr{L}_A(\mathscr{H})^{(p)}$. *Then*

$$\omega_{e,A}^2(\mathbf{X}) \leq \sqrt{\sum_{i=1}^{p} \|X_i\|_A^4 + \sum_{1 \leq i \neq j \leq p} \omega_A^2\big(X_j^{\sharp_A} X_i\big)}.$$

Proof. It follows from Theorem 1 that

$$\begin{aligned} \omega_{e,A}^4(\mathbf{X}) &\leq \left\| \sum_{i=1}^{p} \left(X_i^{\sharp_A} X_i\right)^2 \right\|_A + \sum_{1 \leq i \neq j \leq p} \omega_A^2\left(X_j^{\sharp_A} X_i\right) \\ &\leq \sum_{i=1}^{p} \left\| (X_i^{\sharp_A} X_i)^2 \right\|_A + \sum_{1 \leq i \neq j \leq p} \omega_A^2\left(X_j^{\sharp_A} X_i\right) \\ &= \sum_{i=1}^{p} \left\| X_i^{\sharp_A} X_i \right\|_A^2 + \sum_{1 \leq i \neq j \leq p} \omega_A^2\left(X_j^{\sharp_A} X_i\right), \end{aligned}$$

where the last equality follows by applying (7) since $X_i^{\sharp_A} X_i$ is an A-self-adjoint operator for all $i \in \{1, \ldots, p\}$. Hence, we reach the desired inequality by taking (8) into account. □

We can obtain another significant implication of Theorem 1 by deriving a sharp upper bound for the classical A-numerical radius. This finding enhances our understanding of the A-numerical radius under various conditions.

Corollary 2. *Let* $X \in \mathscr{L}_A(\mathscr{H})$. *Then,*

$$\omega_A^2(X) \leq \frac{1}{4}\sqrt{\left\|(X + X^{\sharp_A})^4 + (X - X^{\sharp_A})^4\right\|_A + 2\omega_A^2\Big[(X^{\sharp_A} - X)(X^{\sharp_A} + X)\Big]}.$$

Moreover, the above inequality is sharp.

Proof. Let $X \in \mathscr{L}_A(\mathscr{H})$. Since $X = \Re_A(X) + i\Im_A(X)$, then we deduce that $X^{\sharp_A} = [\Re_A(X)]^{\sharp_A} - i[\Im_A(X)]^{\sharp_A}$. Further, one observes that

$$\begin{aligned} \omega_A^2(X^{\sharp_A}) &= \sup_{\xi \in S_1^A} \left|\langle X^{\sharp_A}\xi, \xi\rangle_A\right|^2 \\ &= \sup_{\xi \in S_1^A} \left(\left|\langle [\Re_A(X)]^{\sharp_A}\xi, \xi\rangle_A\right|^2 + \left|\langle [\Im_A(X)]^{\sharp_A}\xi, \xi\rangle_A\right|^2 \right) \\ &= \omega_{e,A}^2\Big([\Re_A(X)]^{\sharp_A}, [\Im_A(X)]^{\sharp_A}\Big). \end{aligned}$$

This immediately yields that

$$\omega_A(X) = \omega_{e,A}\Big([\Re_A(X)]^{\sharp_A}, [\Im_A(X)]^{\sharp_A}\Big). \tag{13}$$

On the other hand, by letting $d = 2$ in the second inequality of Theorem 1, we infer that

$$\omega^4_{e,A}(X_1, X_2) \leq \left\| \left(X_1^{\sharp_A} X_1\right)^2 + \left(X_2^{\sharp_A} X_2\right)^2 \right\|_A + \omega_A^2\left(X_1^{\sharp_A} X_2\right) + \omega_A^2\left(X_2^{\sharp_A} X_1\right).$$

By considering both (2) and (3), it becomes clear that

$$\omega^4_{e,A}(X_1, X_2) \leq \left\| \left(X_1^{\sharp_A} X_1\right)^2 + \left(X_2^{\sharp_A} X_2\right)^2 \right\|_A + 2\omega_A^2\left(X_1^{\sharp_A} X_2\right), \tag{14}$$

for any $X_1, X_2 \in \mathscr{L}_A(\mathscr{H})$. Now, let $X \in \mathscr{L}_A(\mathscr{H})$. By using (13) and then applying equality (14) with $X_1 = [\Re_A(X)]^{\sharp_A}$ and $X_2 = [\Im_A(X)]^{\sharp_A}$, we have

$$\begin{aligned}
\omega_A^4(X) &= \omega^4_{e,A}\left([\Re_A(X)]^{\sharp_A}, [\Im_A(X)]^{\sharp_A}\right) \\
&\leq \left\| \left(([\Re_A(X)]^{\sharp_A})^{\sharp_A}[\Re_A(X)]^{\sharp_A}\right)^2 + \left(([\Im_A(X)]^{\sharp_A})^{\sharp_A}[\Im_A(X)]^{\sharp_A}\right)^2 \right\|_A \\
&\quad + 2\omega_A^2\left(([\Re_A(X)]^{\sharp_A})^{\sharp_A}[\Im_A(X)]^{\sharp_A}\right).
\end{aligned}$$

Furthermore, it may be checked that $\Re_A(X)$ and $\Im_A(X)$ are two A-self-adjoint operators. Thus, in view of (4), we have

$$([\Re_A(X)]^{\sharp_A})^{\sharp_A} = [\Re_A(X)]^{\sharp_A} \quad \text{and} \quad ([\Im_A(X)]^{\sharp_A})^{\sharp_A} = [\Im_A(X)]^{\sharp_A}. \tag{15}$$

Taking (15) into consideration, we have

$$\omega_A^4(X) \leq \left\| \left(([\Re_A(X)]^{\sharp_A})^2\right)^2 + \left(([\Im_A(X)]^{\sharp_A})^2\right)^2 \right\|_A + 2\omega_A^2\left([\Re_A(X)]^{\sharp_A}[\Im_A(X)]^{\sharp_A}\right),$$

whence

$$\omega_A^4(X) \leq \left\| ([\Re_A(X)]^4)^{\sharp_A} + ([\Im_A(X)]^4)^{\sharp_A} \right\|_A + 2\omega_A^2\left([\Im_A(X)][\Re_A(X)]\right),$$

where, in the inequality, we use the fact that $\omega_A(T^{\sharp_A}) = \omega_A(T)$ for all $T \in \mathscr{L}_A(\mathscr{H})$. Thus, we have

$$\omega_A^4(X) \leq \left\| [\Re_A(X)]^4 + [\Im_A(X)]^4 \right\|_A + 2\omega_A^2\left([\Im_A(X)][\Re_A(X)]\right).$$

This immediately shows the desired result.

To prove that the inequality in Corollary 2 is sharp, we consider an A-self-adjoint operator T on $\mathscr{H}$. If we choose $X = T^{\sharp_A}$ in Corollary 2 and then apply (4), we have

$$\begin{aligned}
&\left\| \left(T^{\sharp_A} + (T^{\sharp_A})^{\sharp_A}\right)^4 + \left(T^{\sharp_A} - (T^{\sharp_A})^{\sharp_A}\right)^4 \right\|_A + 2\omega_A^2\left[\left((T^{\sharp_A})^{\sharp_A} - T^{\sharp_A}\right)\left((T^{\sharp_A})^{\sharp_A} + T^{\sharp_A}\right)\right] \\
&= \left\| \left(2T^{\sharp_A}\right)^4 \right\|_A = 16\left\| T^{\sharp_A} \right\|_A^4,
\end{aligned}$$

where, in the last part, we used equality (7) since $T^{\sharp_A}$ is also an A-self-adjoint operator. Further, by (6), we have $\omega_A(T^{\sharp_A}) = \|T^{\sharp_A}\|_A$. Hence, we infer that both sides of the inequality in Corollary 2 become $\|T^{\sharp_A}\|_A$. □

Additionally, Theorem 3 has a third application, which is presented in the following corollary.

Corollary 3. *Let $X \in \mathscr{L}_A(\mathscr{H})$. Then,*

$$\omega_A^4(X) \leq \frac{1}{4}\left\| (XX^{\sharp_A})^2 + (X^{\sharp_A}X)^2 \right\|_A + \frac{1}{2}\omega_A^2(X^2). \tag{16}$$

Moreover, inequality (16) *is sharp.*

Proof. Let $X \in \mathscr{L}_A(\mathscr{H})$. We observe that

$$\begin{aligned}\omega_{e,A}(X, X^{\sharp_A}) &= \sup_{\xi \in \mathbb{S}_1^A} \sqrt{\left|\langle X\xi, \xi\rangle_A\right|^2 + \left|\langle X^{\sharp_A}\xi, \xi\rangle_A\right|^2} \\ &= \sqrt{2}\omega_A(X).\end{aligned}$$

This implies that

$$\omega_A^4(X) = \frac{1}{4}\omega_{e,A}^4\left(X^{\sharp_A}, (X^{\sharp_A})^{\sharp_A}\right). \tag{17}$$

Therefore, if we replace X_1 and X_2 in (14) with $X^{\sharp_A}$ and $(X^{\sharp_A})^{\sharp_A}$, respectively, and then we make use of (2) and (17), we have

$$\begin{aligned}\omega_A^4(X) &= \frac{1}{4}\omega_{e,A}^4\left(X^{\sharp_A}, (X^{\sharp_A})^{\sharp_A}\right) \\ &\leq \frac{1}{4}\left[\left\|\left((XX^{\sharp_A})^{\sharp_A}\right)^2 + \left((X^{\sharp_A}X)^{\sharp_A}\right)^2\right\|_A + 2\omega_A^2\left(\left[(X^{\sharp_A})^{\sharp_A}\right]^2\right)\right].\end{aligned}$$

By using the fact that $\|T^{\sharp_A}\|_A = \|T\|_A$ and $\omega_A(T^{\sharp_A}) = \omega_A(T)$ for all $T \in \mathscr{L}_A(\mathscr{H})$, we immediately deduce that

$$\omega_A^4(X) \leq \frac{1}{4}\left(\left\|(XX^{\sharp_A})^2 + (X^{\sharp_A}X)^2\right\|_A + 2\omega_A^2(X^2)\right).$$

Hence, we obtain the desired inequality (16). To prove the sharpness of inequality (16), we consider an A-normal operator S. By [17], we have S^2, which is also A-normal. Furthermore, in view of [14], we deduce that the following properties

$$\omega_A(T) = \|T\|_A \quad , \omega_A(T^n) = \omega_A^n(T) \quad \text{and} \quad \|T^n\|_A = \|T\|_A^n, \quad \forall\, n \in \mathbb{N}^* \tag{18}$$

hold for any A-normal operator T. Thus, by using (18), we see that

$$\begin{aligned}\frac{1}{4}\left\|(SS^{\sharp_A})^2 + (S^{\sharp_A}S)^2\right\|_A + \frac{1}{2}\omega_A^2(S^2) &= \frac{1}{2}\left\|(S^{\sharp_A}S)^2\right\|_A + \frac{1}{2}\omega_A^4(S) \\ &= \frac{1}{2}\left\|S^{\sharp_A}S\right\|_A^2 + \frac{1}{2}\|S\|_A^4 \\ &= \frac{1}{2}\|S\|_A^4 + \frac{1}{2}\|S\|_A^4 \\ &= \|S\|_A^4 = \omega_A(S).\end{aligned}$$

Therefore, the desired results are achieved. □

The following lemma will be useful in proving our next result. To prove this lemma, we apply the Boas–Bellman type inequality established by the second author (see [26]) and use the same argument as in the proof of Lemma 1.

Lemma 2. *Let $y_1, \ldots, y_p$ be vectors in $\mathscr{H}$. Then, for all $x \in \mathscr{H}$, we have*

$$\sum_{i=1}^{p}\left|\langle x, y_m\rangle_A\right|^2 \leq \|x\|_A^2\left(\max_{1\leq m\leq p}\|y_m\|_A^2 + (p-1)\max_{1\leq m\neq k\leq p}\left|\langle y_m, y_k\rangle_A\right|\right).$$

Our preparation has led us to achieve the following outcome:

Theorem 2. *Let* $\mathbf{X} = (X_1, \ldots, X_p) \in \mathcal{L}_A(\mathcal{H})^{(p)}$. *Then*

$$\omega_{e,A}(\mathbf{X}) \leq \sqrt{\max_{1\leq m\leq p} \|X_m\|_A^2 + (p-1) \max_{1\leq m\neq k\leq p} \omega_A\big(X_k^{\sharp_A} X_m\big)}.$$

Proof. Let $x \in S_1^A$. By letting $x = \xi$ and $y_m = X_m\xi$, for all $m \in \{1, \ldots, p\}$ in Lemma 2, we have

$$\begin{aligned}
&\sum_{m=1}^{p} \left|\langle X_m\xi, \xi\rangle_A\right|^2 \\
&\leq \|\xi\|_A^2 \left[\max_{1\leq m\leq p} \|X_m\xi\|_A^2 + (p-1) \max_{1\leq m\neq k\leq p} \left|\langle X_m\xi, X_k\xi\rangle_A\right|\right] \\
&\leq \|\xi\|_A^2 \left[\max_{1\leq m\leq p} \|X_m\xi\|_A^2 + (p-1) \max_{1\leq m\neq k\leq p} \left|\langle X_k^{\sharp_A} X_m\xi, \xi\rangle_A\right|\right] \\
&\leq \|\xi\|_A^2 \left[\sup_{\xi\in S_1^A} \left(\max_{1\leq m\leq p} \|X_m\xi\|_A^2\right) + (p-1) \sup_{\xi\in S_1^A} \left(\max_{1\leq m\neq k\leq p} \left|\langle X_k^{\sharp_A} X_m\xi, \xi\rangle_A\right|\right)\right] \\
&\leq \|\xi\|_A^2 \left[\max_{1\leq m\leq p} \|X_m\|_A^2 + (p-1) \max_{1\leq m\neq k\leq p} \omega_A\big(X_k^{\sharp_A} X_m\big)\right].
\end{aligned}$$

Taking the supremum over all $\xi \in S_1^A$ in the last inequality, we have

$$\omega_{e,A}^2(\mathbf{X}) \leq \left[\max_{1\leq m\leq p} \|X_m\|_A^2 + (p-1) \max_{1\leq m\neq k\leq p} \omega_A\big(X_k^{\sharp_A} X_m\big)\right].$$

Hence, we have reached the desired inequality. □

Remark 1. *(1) If we set $p = 2$ in Theorem 2, a recent result established in [16] can be obtained. This result provides sharp inequalities for any $X_1, X_2 \in \mathcal{L}_A(\mathcal{H})$, given by:*

$$\omega_{e,A}(X_1, X_2) \leq \sqrt{\max\big(\|X_1\|_A^2, \|X_2\|_A^2\big) + \omega_A\big(X_2^{\sharp_A} X_1\big)}. \tag{19}$$

(2) Theorem 2.5 in [24] can be derived as a special case of Theorem 2 when weight A is chosen to be the identity operator I.

Moving forward, we introduce a natural generalization of the widely recognized Boas–Bellman inequality (refer to [27–29] (Section 4) for more information) in the following lemma. The proof follows a similar approach as the previous one and will be skipped.

Lemma 3. *Let $y_1, \ldots, y_p$ be vectors in $\mathcal{H}$. Then, for all $x \in \mathcal{H}$, we have*

$$\sum_{m=1}^{p} \left|\langle x, y_m\rangle_A\right|^2 \leq \|x\|_A^2 \left[\max_{m\in\{1,\ldots,p\}} \|y_m\|_A^2 + \left(\sum_{1\leq m\neq k\leq p} \left|\langle y_m, y_k\rangle_A\right|^2\right)^{\frac{1}{2}}\right].$$

The theorem below introduces a new upper bound for the joint A-numerical radius of operators that have A-adjoint operators.

Theorem 3. *Let* $\mathbf{X} = (X_1, \ldots, X_p) \in \mathcal{L}_A(\mathcal{H})^{(p)}$, *then*

$$\omega_{e,A}(\mathbf{X}) \leq \left(\max_{m\in\{1,\ldots,p\}} \|X_m\|_A^2 + \left[\sum_{1\leq m\neq k\leq p} \omega_A^2\big(X_k^{\sharp_A} X_m\big)\right]^{\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{20}$$

Proof. Let $x \in S_1^A$. By letting $x = \xi$ and $y_m = X_m\xi$, for all $m \in \{1, \ldots, p\}$ in Lemma 3, we have

$$\begin{aligned}\sum_{m=1}^{p}\left|\langle X_m\xi, \xi\rangle_A\right|^2 &\leq \max_{m\in\{1,\ldots,p\}}\|X_m\xi\|_A^2 + \left(\sum_{1\leq m\neq k\leq p}\left|\langle X_k^{\sharp_A}X_m\xi, \xi\rangle_A\right|^2\right)^{\frac{1}{2}} \\ &\leq \sup_{\xi\in S_1^A}\left[\max_{m\in\{1,\ldots,p\}}\|X_m\xi\|_A^2\right] + \sup_{\xi\in S_1^A}\left(\sum_{1\leq m\neq k\leq p}\left|\langle X_k^{\sharp_A}X_m\xi, \xi\rangle_A\right|^2\right)^{\frac{1}{2}} \\ &\leq \max_{m\in\{1,\ldots,p\}}\|X_m\|_A^2 + \left(\sum_{1\leq m\neq k\leq p}\omega_A^2(X_k^{\sharp_A}X_m)\right)^{\frac{1}{2}}.\end{aligned}$$

Taking the supremum over all $\xi \in S_1^A$ in the last inequality, we have

$$\omega_{e,A}^2(\mathbf{X}) \leq \max_{m\in\{1,\ldots,p\}}\|X_m\|_A^2 + \left(\sum_{1\leq m\neq k\leq p}\omega_A^2(X_k^{\sharp_A}X_m)\right)^{\frac{1}{2}}.$$

Hence, we have reached the desired inequality. □

Remark 2. *Theorem 3 provides a new upper bound for the joint A-numerical radius of operators X_1 and X_2 that have A-adjoints. Setting $p = 2$ in this theorem yields the inequality*

$$\omega_{e,A}(X_1, X_2) \leq \sqrt{\max(\|X_1\|_A^2, \|X_2\|_A^2) + \sqrt{2}\omega_A(X_2^{\sharp_A}X_1)}, \tag{21}$$

which is valid for all $X_1, X_2 \in \mathscr{L}_A(\mathscr{H})$. However, it is important to note that inequality (19) obtained from Theorem 2 is sharper than (21). This highlights the importance of Theorem 2 in producing more accurate estimates for the A-joint numerical radius of semi-Hilbert space operators.

We can establish the following useful lemma by utilizing a Boas–Bellman type inequality, which is well-known and was proven in [29] (p. 132) (also refer to [26]).

Lemma 4. *Let $y_1, \ldots, y_p$ be vectors in $\mathscr{H}$. For all $x \in \mathscr{H}$, we have*

$$\sum_{i=1}^{p}\left|\langle x, y_i\rangle_A\right|^2 \leq \|x\|_A \max_{1\leq i\leq p}\left|\langle x, y_i\rangle_A\right|\sqrt{\sum_{i=1}^{p}\|y_i\|_A^2 + \sum_{1\leq i\neq j\leq p}\left|\langle y_i, y_j\rangle_A\right|}.$$

Using the above lemma, we can derive the following result.

Theorem 4. *Let $\mathbf{X} = (X_1, \ldots, X_p) \in \mathscr{L}_A(\mathscr{H})^{(p)}$, then*

$$\omega_{e,A}^2(\mathbf{X}) \leq \max_{1\leq i\leq p}\omega_A(X_i)\sqrt{\left\|\sum_{i=1}^{p}X_i^{\sharp_A}X_i\right\|_A + \sum_{1\leq i\neq j\leq p}\omega_A\left(X_j^{\sharp_A}X_i\right)}. \tag{22}$$

In particular, if $AX_j^{\sharp_A}X_i = 0$ for all $i, j \in \{1, \ldots, p\}$ with $i \neq j$, then

$$\omega_{e,A}(\mathbf{X}) \leq \sqrt{\max_{1\leq i\leq p}\omega_A(X_i)}\,\|\mathbf{X}\|_A. \tag{23}$$

Proof. Let $\xi \in S_1^A$. By letting $x = \xi$ and $y_k = X_k\xi$ for all $k \in \{1, \dots, p\}$ in Lemma 4, we see that

$$\begin{aligned}
\sum_{i=1}^{p} \left|\langle X_i\xi, \xi\rangle_A\right|^2 &\leq \max_{1\leq i\leq p} \left|\langle X_i\xi, \xi\rangle_A\right| \sqrt{\sum_{i=1}^{p} \|X_i\xi\|_A^2 + \sum_{1\leq i\neq j\leq p} \left|\langle X_i\xi, X_j\xi\rangle_A\right|} \\
&= \max_{1\leq i\leq p} \left|\langle X_i\xi, \xi\rangle_A\right| \sqrt{\sum_{i=1}^{p} \|X_i\xi\|_A^2 + \sum_{1\leq i\neq j\leq p} \left|\langle X_j^{\sharp_A} X_i\xi, \xi\rangle_A\right|} \\
&\leq \max_{1\leq i\leq p} \omega_A(X_i) \sqrt{\|\mathbf{X}\|_A^2 + \sum_{1\leq i\neq j\leq p} \omega_A\big(X_j^{\sharp_A} X_i\big)}.
\end{aligned}$$

By taking the supremum over all $\xi \in S_1^A$, we have

$$\omega_{e,A}^2(\mathbf{X}) \leq \max_{1\leq i\leq p} \omega_A(X_i) \sqrt{\|\mathbf{X}\|_A^2 + \sum_{1\leq i\neq j\leq p} \omega_A\big(X_j^{\sharp_A} X_i\big)}. \tag{24}$$

Therefore, the desired inequality (22) is achieved by applying (9). Finally, since $AX_j^{\sharp_A} X_i = 0$ for all $i, j \in \{1, \dots, p\}$ with $i \neq j$, then $\omega_A\big(X_j^{\sharp_A} X_i\big) = 0$ for every $i, j \in \{1, \dots, p\}$ with $i \neq j$, and inequality (23) is achieved by taking (24) into account. This completes our proof. □

Remark 3. *By letting $p = 2$ in Theorem 4, we obtain a recent result proved in [16]. Namely, for every $X_1, X_2 \in \mathscr{L}_A(\mathscr{H})$, we have*

$$\omega_{e,A}(X_1, X_2) \leq \sqrt{\max\Big\{\omega_A(X_1), \omega_A(X_2)\Big\} \sqrt{\big\|X_1^{\sharp_A} X_1 + X_2^{\sharp_A} X_2\big\|_A + 2\omega_A\big(X_2^{\sharp_A} X_1\big)}}.$$

If we apply (10) for $p = 1$, we have

$$\omega_{e,A}(X_1, X_2) \leq \sqrt{\max\Big\{\|X_1\|_A, \|X_2\|_A\Big\} \sqrt{\big\|X_1^{\sharp_A} X_1 + X_2^{\sharp_A} X_2\big\|_A + 2\omega_A\big(X_2^{\sharp_A} X_1\big)}}. \tag{25}$$

The following corollary provides an upper bound for $\omega_A(\cdot)$ using (25), which follows as an application of the previous result.

Corollary 4. *Let $X \in \mathscr{L}_A(\mathscr{H})$. Then*

$$\omega_A^2(X) \leq \frac{\sqrt{2}}{4} \max\Big\{\gamma_A(X), \Gamma_A(X)\Big\} \sqrt{\big\|X^{\sharp_A} X + XX^{\sharp_A}\big\|_A + \omega_A\Big((X + X^{\sharp_A})(X - X^{\sharp_A})\Big)},$$

where $\gamma_A(X) = \big\|X + X^{\sharp_A}\big\|_A$ and $\Gamma_A(X) = \big\|X - X^{\sharp_A}\big\|_A$. Moreover, the above inequality is sharp.

Proof. Let $X \in \mathscr{L}_A(\mathscr{H})$. First, note that a short calculation shows that

$$\Big([\Re_A(X)]^{\sharp_A}\Big)^2 + \Big([\Im_A(X)]^{\sharp_A}\Big)^2 = \left(\frac{XX^{\sharp_A} + X^{\sharp_A} X}{2}\right)^{\sharp_A}. \tag{26}$$

By applying (25) for $X_1 = [\Re_A(X)]^{\sharp_A}$ and $X_2 = [\Im_A(X)]^{\sharp_A}$ and then using (15) together with (13), we observe that

$$\omega_A^2(X) \leq \max\Big\{\big\|[\Re_A(X)]^{\sharp_A}\big\|_A, \big\|[\Im_A(X)]^{\sharp_A}\big\|_A\Big\} \zeta_A(X),$$

where

$$\zeta_A(X) = \sqrt{\left\| \left([\Re_A(X)]^{\sharp_A}\right)^2 + \left([\Im_A(X)]^{\sharp_A}\right)^2 \right\|_A + 2\omega_A\left([\Im_A(X)]^{\sharp_A}[\Re_A(X)]^{\sharp_A}\right)}.$$

This implies that

$$\begin{aligned}\omega_A^2(X) &\leq \max\left\{\|\Re_A(X)\|_A, \|\Im_A(X)\|_A\right\}\zeta_A(X) \\ &= \frac{1}{2}\max\left\{\|X+X^{\sharp_A}\|_A, \|X-X^{\sharp_A}\|_A\right\}\zeta_A(X).\end{aligned}$$

On the other hand, by using (26), we see that

$$\begin{aligned}\zeta_A(X) &= \sqrt{\frac{1}{2}\left\|\left(XX^{\sharp_A}+X^{\sharp_A}X\right)^{\sharp_A}\right\|_A + 2\omega_A\left(\Re_A(X)\Im_A(X)\right)} \\ &= \sqrt{\frac{1}{2}\|XX^{\sharp_A}+X^{\sharp_A}X\|_A + \frac{1}{2}\omega_A\left((X+X^{\sharp_A})(X-X^{\sharp_A})\right)} \\ &= \frac{\sqrt{2}}{2}\sqrt{\|XX^{\sharp_A}+X^{\sharp_A}X\|_A + \omega_A\left((X+X^{\sharp_A})(X-X^{\sharp_A})\right)}.\end{aligned}$$

The sharpness of the given inequality can be demonstrated by considering any A-self-adjoint operator T and applying the same approach as in Corollary 2. □

We now state a lemma that can be proved using the Bombieri inequality (see [30] (p. 394), [31], or [29] (p. 134)), along with a similar argument to the one used in the proof of Lemma 1. The statement of the lemma is as follows:

Lemma 5. *Let $y_1, \ldots, y_p$ be vectors in $\mathscr{H}$. Then, for all $x \in \mathscr{H}$, we have*

$$\sum_{i=1}^{p}\left|\langle x, y_i\rangle_A\right|^2 \leq \|x\|_A^2 \max_{1\leq i\leq p}\left\{\sum_{j=1}^{p}\left|\langle y_i, y_j\rangle_A\right|\right\}. \tag{27}$$

Our next result is as follows (and we will provide a proof for it now):

Theorem 5. *Let $\mathbf{X} = (X_1, \ldots, X_p) \in \mathscr{L}_A(\mathscr{H})^{(p)}$. Then*

$$\omega_{e,A}^2(\mathbf{X}) \leq \max_{1\leq i\leq p}\left\{\sum_{j=1}^{p}\omega_A\left(X_j^{\sharp_A}X_i\right)\right\}. \tag{28}$$

Proof. Let $\xi \in S_1^A$. By applying (27) for $x = \xi$ and $y_m = X_m\xi$ for all $m \in \{1, \ldots, p\}$, we see that

$$\begin{aligned}\sum_{i=1}^{p}\left|\langle X_i\xi, \xi\rangle_A\right|^2 &\leq \max_{1\leq i\leq p}\left\{\sum_{j=1}^{p}\left|\langle X_i\xi, X_j\xi\rangle_A\right|\right\} \\ &\leq \max_{1\leq i\leq p}\left\{\sum_{j=1}^{p}\left|\langle X_j^{\sharp_A}X_i\xi, \xi\rangle_A\right|\right\} \\ &\leq \max_{1\leq i\leq p}\left\{\sum_{j=1}^{p}\omega_A\left(X_j^{\sharp_A}X_i\right)\right\}.\end{aligned}$$

By taking the supremum over all $\xi \in S_1^A$ in the last inequality, we reach the desired inequality. □

Remark 4. *By letting $p = 2$ in Theorem 5, we deduce that for every $X_1, X_2 \in \mathscr{L}_A(\mathscr{H})$, we have*

$$\omega^2_{e,A}(X_1, X_2) \leq \max\Big\{\omega_A(X_1^{\sharp_A} X_1) + \omega_A(X_2^{\sharp_A} X_1), \omega_A(X_1^{\sharp_A} X_2) + \omega_A(X_2^{\sharp_A} X_2)\Big\}.$$

By applying the second inequality in (10) for $p = 1$, together with (8), we have

$$\omega_A(X_1^{\sharp_A} X_1) \leq \|X_1\|_A^2 \quad \text{and} \quad \omega_A(X_2^{\sharp_A} X_2) \leq \|X_2\|_A^2.$$

Hence, we have

$$\omega^2_{e,A}(X_1, X_2) \leq \max\Big\{\|X_1\|_A^2 + \omega_A(X_2^{\sharp_A} X_1), \omega_A(X_1^{\sharp_A} X_2) + \|X_2\|_A^2\Big\}.$$

On the other hand, by applying (3), we see that

$$\begin{aligned}\omega_A(X_2^{\sharp_A} X_1) &= \omega_A(X_1^{\sharp_A} P_{\overline{\mathcal{R}(A)}} X_2 P_{\overline{\mathcal{R}(A)}}) \\ &= \omega_A(X_1^{\sharp_A} X_2 P_{\overline{\mathcal{R}(A)}}) = \omega_A(X_1^{\sharp_A} X_2).\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}\omega^2_{e,A}(X_1, X_2) &\leq \max\Big\{\|X_1\|_A^2 + \omega_A(X_2^{\sharp_A} X_1), \omega_A(X_1^{\sharp_A} X_2) + \|X_2\|_A^2\Big\} \\ &= \max\Big\{\|X_1\|_A^2 + \omega_A(X_2^{\sharp_A} X_1), \omega_A(X_2^{\sharp_A} X_1) + \|X_2\|_A^2\Big\},\end{aligned}$$

whence

$$\omega^2_{e,A}(X_1, X_2) = \max\Big\{\|X_1\|_A^2, \|X_2\|_A^2\Big\} + \omega_A(X_2^{\sharp_A} X_1).$$

Therefore, we obtain inequality (19).

We can easily derive the following lemma by applying a result proved by the second author in [24] and using the same argument as above.

Lemma 6. *Let $y_1, \ldots, y_p$ be vectors in $\mathscr{H}$. Then, for all $x \in \mathscr{H}$, we have*

$$\sum_{i=1}^{p} \Big|\langle x, y_i\rangle_A\Big|^2 \leq \|x\|_A \min\Big\{\widetilde{\Gamma_A}, \widetilde{\gamma_A}, \widetilde{\delta_A}\Big\},$$

where

$$\widetilde{\Gamma_A} := \begin{cases} \max\limits_{k\in\{1,\ldots,p\}} \Big|\langle x, y_k\rangle_A\Big| \left(\sum\limits_{i,j=1}^{p} \Big|\langle y_i, y_j\rangle_A\Big|\right)^{\frac{1}{2}}; \\ \text{or} \\ \max\limits_{k\in\{1,\ldots,p\}} \Big|\langle x, y_k\rangle_A\Big|^{\frac{1}{2}} \left(\sum\limits_{i=1}^{p} \Big|\langle x, y_i\rangle_A\Big|^r\right)^{\frac{1}{2r}} \left[\sum\limits_{i=1}^{p}\left(\sum\limits_{j=1}^{p} \Big|\langle y_i, y_j\rangle_A\Big|\right)^s\right]^{\frac{1}{2s}}, \\ \qquad \text{where } r, s > 1 \text{ and } \frac{1}{r} + \frac{1}{s} = 1; \\ \text{or} \\ \max\limits_{k\in\{1,\ldots,p\}} \Big|\langle x, y_k\rangle_A\Big|^{\frac{1}{2}} \left(\sum\limits_{i=1}^{p} \Big|\langle x, y_i\rangle_A\Big|\right)^{\frac{1}{2}} \max\limits_{i\in\{1,\ldots,p\}} \left[\sum\limits_{j=1}^{p} \Big|\langle y_i, y_j\rangle_A\Big|\right]^{\frac{1}{2}}; \end{cases}$$

$$\widetilde{\gamma_A} := \begin{cases} \left(\sum_{k=1}^{p}\left|\langle x, y_k\rangle_A\right|^l\right)^{\frac{1}{2l}} \max_{i\in\{1,\dots,p\}}\left|\langle x, y_i\rangle_A\right|^{\frac{1}{2}}\left[\sum_{i=1}^{p}\left(\sum_{j=1}^{p}\left|\langle y_i, y_j\rangle_A\right|\right)^m\right]^{\frac{1}{2m}}, \\ \qquad \text{where } l > 1 \text{ and } \frac{1}{l}+\frac{1}{m}=1; \\ or \\ \left(\sum_{k=1}^{p}\left|\langle x, y_k\rangle_A\right|^l\right)^{\frac{1}{2l}}\left(\sum_{i=1}^{p}\left|\langle x, y_i\rangle_A\right|^t\right)^{\frac{1}{2t}}\left[\sum_{i=1}^{p}\left(\sum_{j=1}^{p}\left|\langle y_i, y_j\rangle_A\right|^m\right)^{\frac{u}{m}}\right]^{\frac{1}{2u}}, \\ \qquad \text{where } l > 1,\ \frac{1}{l}+\frac{1}{m}=1 \text{ and } \frac{1}{t}+\frac{1}{u}=1 \text{ for } t>1; \\ or \\ \left(\sum_{k=1}^{p}\left|\langle x, y_k\rangle_A\right|^l\right)^{\frac{1}{2l}}\left(\sum_{i=1}^{p}\left|\langle x, y_i\rangle_A\right|\right)^{\frac{1}{2}} \max_{i\in\{1,\dots,p\}}\left\{\left(\sum_{j=1}^{p}\left|\langle y_i, y_j\rangle_A\right|^m\right)^{\frac{1}{m}}\right\}, \\ \qquad \text{where } l > 1 \text{ and } \frac{1}{l}+\frac{1}{m}=1; \end{cases}$$

and

$$\widetilde{\delta_A} := \begin{cases} \left(\sum_{k=1}^{p}\left|\langle x, y_k\rangle_A\right|\right)^{\frac{1}{2}} \max_{i\in\{1,\dots,p\}}\left|\langle x, y_i\rangle_A\right|^{\frac{1}{2}}\sum_{i=1}^{p}\left[\max_{j\in\{1,\dots,p\}}\left|\langle y_i, y_j\rangle_A\right|\right]^{\frac{1}{2}}; \\ or \\ \left(\sum_{k=1}^{p}\left|\langle x, y_k\rangle_A\right|\right)^{\frac{1}{2}}\left(\sum_{i=1}^{p}\left|\langle x, y_i\rangle_A\right|^m\right)^{\frac{1}{2m}}\left[\sum_{i=1}^{p}\left[\max_{j\in\{1,\dots,p\}}\left|\langle y_i, y_j\rangle_A\right|^l\right]\right]^{\frac{1}{2l}}, \\ \qquad \text{where } m > 1 \text{ and } \frac{1}{m}+\frac{1}{l}=1; \\ or \\ \sum_{k=1}^{p}\left|\langle x, y_k\rangle_A\right| \max_{i,j\in\{1,\dots,p\}}\left|\langle y_i, y_j\rangle_A\right|^{\frac{1}{2}}. \end{cases}$$

An upper bound for $\omega_A(\cdot)$ can be obtained by applying Lemma 6. The resulting bound is stated as follows.

Theorem 6. *Let* $\mathbf{X} = (X_1, \dots, X_p) \in \mathscr{L}_A(\mathscr{H})^{(p)}$. *Then*

$$\omega_{e,A}^2(\mathbf{X}) \le \min\left\{\Gamma_A, \gamma_A, \delta_A\right\},$$

where

$$\Gamma_A := \begin{cases} \max_{k\in\{1,\dots,p\}}\left\{\omega_A(X_k)\right\}\sqrt{\sum_{i,j=1}^{p}\omega_A\left(X_j^{\sharp_A}X_i\right)}; \\ or \\ \max_{k\in\{1,\dots,p\}}\left(\sqrt{\omega_A(X_k)}\right)\left(\sum_{i=1}^{p}\left[\omega_A(X_i)\right]^r\right)^{\frac{1}{2r}}\left[\sum_{i=1}^{p}\left(\sum_{j=1}^{p}\omega_A\left(X_j^{\sharp_A}X_i\right)\right)^s\right]^{\frac{1}{2s}}, \\ \qquad \text{where } r, s > 1 \text{ and } \frac{1}{r}+\frac{1}{s}=1; \\ or \\ \max_{k\in\{1,\dots,p\}}\left(\sqrt{\omega_A(X_k)}\right)\sqrt{\sum_{i=1}^{p}\omega_A(X_i)} \max_{i\in\{1,\dots,p\}}\left(\sqrt{\sum_{j=1}^{p}\omega_A\left(X_j^{\sharp_A}X_i\right)}\right); \end{cases}$$

$$\gamma_A := \begin{cases} \left(\sum_{k=1}^{p}[\omega_A(X_k)]^l\right)^{\frac{1}{2l}} \max_{k\in\{1,\dots,p\}}\left(\sqrt{\omega_A(X_k)}\right)\left[\sum_{i=1}^{p}\left(\sum_{j=1}^{p}\omega_A(X_j^{\sharp_A}X_i)\right)^m\right]^{\frac{1}{2m}}, \\ \qquad \text{where } l>1 \text{ and } \frac{1}{l}+\frac{1}{m}=1; \\ \text{or} \\ \left(\sum_{k=1}^{p}\left[\omega_A(X_k)\right]^l\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{p}\left[\omega_A(X_i)\right]^t\right)^{\frac{1}{2t}}\left[\sum_{i=1}^{p}\left(\sum_{j=1}^{p}\left[\omega_A(X_j^{\sharp_A}X_i)\right]^m\right)^{\frac{u}{m}}\right]^{\frac{1}{2u}}, \\ \qquad \text{where } l,t>1,\ \frac{1}{l}+\frac{1}{m}=1 \text{ and } \frac{1}{t}+\frac{1}{u}=1; \\ \text{or} \\ \left(\sum_{k=1}^{p}\left[\omega_A(X_k)\right]^l\right)^{\frac{1}{2p}}\sqrt{\sum_{i=1}^{p}\omega_A(X_i)}\max_{i\in\{1,\dots,p\}}\left\{\left(\sum_{j=1}^{p}\left[\omega_A(X_j^{\sharp_A}X_i)\right]^m\right)^{\frac{1}{2m}}\right\}, \\ \qquad \text{where } l>1 \text{ and } \frac{1}{l}+\frac{1}{m}=1; \end{cases}$$

and

$$\delta_A := \begin{cases} \sqrt{\sum_{k=1}^{p}\omega_A(X_k)}\max_{i\in\{1,\dots,p\}}\left(\sqrt{\omega_A(X_i)}\right)\sum_{i=1}^{p}\left[\max_{j\in\{1,\dots,p\}}\left\{\sqrt{\omega_A(X_j^{\sharp_A}X_i)}\right\}\right]; \\ \text{or} \\ \sqrt{\sum_{k=1}^{p}\omega_A(X_k)}\left(\sum_{i=1}^{p}\left[\omega_A(X_i)\right]^m\right)^{\frac{1}{2m}}\sum_{i=1}^{p}\left[\max_{j\in\{1,\dots,p\}}\left[\omega_A(X_j^{\sharp_A}X_i)\right]^l\right]^{\frac{1}{2l}}, \\ \qquad \text{where } m>1 \text{ and } \frac{1}{m}+\frac{1}{l}=1; \\ \text{or} \\ \sum_{k=1}^{p}\omega_A(X_k)\max_{i,j\in\{1,\dots,p\}}\left\{\sqrt{\omega_A(X_j^{\sharp_A}X_i)}\right\}. \end{cases}$$

The above theorem has various practical applications, one of which we will state without proof. This is because the proof employs techniques that have already been utilized in this work.

Corollary 5. *Let $X \in \mathscr{L}_A(\mathscr{H})$. Then*

$$\omega_A^2(X) \le \frac{1}{4}\left(\left\|X+X^{\sharp_A}\right\|_A+\left\|X-X^{\sharp_A}\right\|_A\right)\max\left\{\left\|X+X^{\sharp_A}\right\|,\left\|X-X^{\sharp_A}\right\|,\theta_A(X)\right\},$$

where

$$\theta_A = \sqrt{\omega_A\left(\left(X^{\sharp_A}-X\right)\left(X^{\sharp_A}+X\right)\right)}.$$

The constant $\frac{1}{4}$ is also sharp.

4. Conclusions

In this paper, we made significant progress in the study of p-tuples of bounded linear operators on a complex Hilbert space with adjoint operators defined with respect to a non-zero positive operator A. Our focus was on investigating the joint A-numerical radius of the p-tuple, which was introduced in [3]. Our main contribution was in establishing several upper bounds for the joint A-numerical radius, some of which extended and improved upon previous work [24]. Our results have far-reaching implications beyond the specific context of semi-Hilbert space operators. As an application of our findings, we presented sharp bounds for the classical A-numerical radius. These results not only contribute to the ongoing research in operator theory and functional analysis but will also pave the way for further exploration of this topic. Our work builds upon the recent research presented in [16,24], utilizing extensions of the well-known Bessel inequality developed by Bombieri, the third author, and Boas–Bellman. By combining these results, we were able to derive new insights into the joint A-numerical radius of semi-Hilbert space operators.

Our paper represents a significant advance in the study of operator theory and functional analysis. It has far-reaching implications and could serve as a starting point for future research in this area. One potential avenue for future research is to explore the possibility of extending our results to the study of the joint $\mathbb{A}$-numerical radius for p-tuples of operator matrices with entries belonging to $\mathcal{L}_A(\mathcal{H})$ or are A-bounded operators. This would require deeper exploration to determine if such a generalization is feasible. Moreover, our findings could inspire further investigation into other related topics, such as the joint A-spectral radius and the joint A-numerical range, which may have significant applications.

Since the joint numerical radius has several applications in applied mathematics, we expect to study the applications of the A-joint numerical radius in other sciences. In particular, the A-joint numerical radius may be relevant in the study of quantum mechanics and quantum computing. These applications, however, require further exploration and will be left for future research.

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References

1. Baklouti, H.; Namouri, S. Spectral analysis of bounded operators on semi-Hilbertian spaces. *Banach J. Math. Anal.* **2022**, *16*, 12. [CrossRef]
2. Bhunia, P.; Dragomir, S.S.; Moslehian, M.S.; Paul, K. *Lectures on Numerical Radius Inequalities*; Infosys Science Foundation Series in Mathematical Sciences; Springer: Cham, Switzerland, 2022.
3. Baklouti, H.; Feki, K.; Ahmed, O.A.M.S. Joint numerical ranges of operators in semi-Hilbertian spaces. *Linear Algebra Appl.* **2018**, *555*, 266–284. [CrossRef]
4. Kittaneh, F.; Zamani, A. Bounds for $\mathbb{A}$-numerical radius based on an extension of A-Buzano inequality. *J. Comput. Appl. Math.* **2023**, *426*, 115070. [CrossRef]
5. Kittaneh, F.; Zamani, A. A refinement of A-Buzano inequality and applications to A-numerical radius inequalities. *Linear Algebra Appl.* 2023, *in press*. [CrossRef]
6. Zamani, A. A-Numerical radius inequalities for semi-Hilbertian space operators. *Linear Algebra Appl.* **2019**, *578*, 159–183. [CrossRef]
7. Krejčiřik, D.; Lotoreichik, V.; Znojil, M.The minimally anisotropic metric operator in quasi-Hermitian quantum mechanics. *Proc. R. Soc.* **2018**, *474*, 20180264. [CrossRef] [PubMed]
8. Scholtz, F.G.; Geyer, H.B.; Hahne, F.J.W. Quasi-Hermitian operators in quantum mechanics and the variational principle. *Ann. Phys.* **1992**, *213*, 74–101. [CrossRef]
9. Mostafazadeh, A. Pseudo-Hermitian representation of quantum mechanics. *Int. J. Geom. Methods Mod. Phys.* **2010**, *7*, 1191–1306. [CrossRef]
10. Azizov, T.Y.; Iokhvidov, I.S. *Linear Operators in Spaces with Indefinite Metric*; Dawson, E.R., Ed.; Wiley: Chichester, UK, 1989.
11. Gutkin, E.; Życzkowski, K. Joint numerical ranges, quantum maps, and joint numerical shadows. *Linear Algebra Appl.* **2013**, *438*, 2394–2404. [CrossRef]
12. Müller, V.; Tomilov, Y. Joint numerical ranges: Recent advances and applications minicourse by V. Müller and Y. Tomilov. *Concr. Oper.* **2020**, *7*, 133–154. [CrossRef]
13. Altwaijry, N.; Feki, K.; Minculete, N. A new seminorm for d-tuples of A-bounded operators and its applications. *Mathematics* **2023**, *11*, 685. [CrossRef]

14. Feki, K. Spectral radius of semi-Hilbertian space operators and its applications. *Ann. Funct. Anal.* **2020**, *11*, 929–946. [CrossRef]
15. Feki, K. On tuples of commuting operators in positive semidefinite inner product spaces. *Linear Algebra Appl.* **2020**, *603*, 313–328. [CrossRef]
16. Feki, K. Inequalities for the A-joint numerical radius of two operators and their applications. *Hacettepe J. Math. Stat.* 2023, *early access*.
17. Saddi, A. A-Normal operators in semi-Hilbertian spaces. *Aust. J. Math. Anal. Appl.* **2012**, *9*, 1–12.
18. Bhunia, P.; Paul, K.; Nayak, R.K. On inequalities for A-numerical radius of operators. *Elem. J. Linear Algebra* **2020**, *36*, 143–157.
19. Moslehian, M.S.; Xu, Q.; Zamani, A. Seminorm and numerical radius inequalities of operators in semi-Hilbertian spaces. *Linear Algebra Appl.* **2020**, *591*, 299–321. [CrossRef]
20. Arias, M.L.; Corach, G.; Gonzalez, M.C. Partial isometries in semi-Hilbertian spaces. *Linear Algebra Appl.* **2008**, *428*, 1460–1475. [CrossRef]
21. Douglas, R.G. On majorization, factorization and range inclusion of operators in Hilbert space. *Proc. Am. Math. Soc.* **1966**, *17*, 413–416. [CrossRef]
22. Rout, N.C.; Mishra, D. Further results on A-numerical radius inequalities. *Ann. Funct. Anal.* **2022**, *13*, 13. [CrossRef]
23. Feki, K. A note on the A-numerical radius of operators in semi-Hilbert spaces. *Arch. Math.* **2020**, *115*, 535–544. [CrossRef]
24. Dragomir, S.S. Upper Bounds for the Euclidean Operator Radius and Applications. *J. Inequal. Appl.* **2008**, *2008*, 472146. [CrossRef]
25. Dragomir, S.S.; Mond, B.; Pećarixcx, J. Some remarks on Bessel's innequlity in inner product spaces. *Stud. Univ. Babeş -Bolyai Math.* **1992**, *37*, 77–86.
26. Dragomir, S.S. On the Boas-Bellman inequality in inner product spaces. *Bull. Aust. Math. Soc.* **2004**, *69*, 217–225. [CrossRef]
27. Boas, R.P. A general moment problem. *Am. J. Math.* **1941**, *63*, 361–370. [CrossRef]
28. Bellman, R. Almost orthogonal series. *Bull. Am. Math. Soc.* **1941**, *50*, 517–519. [CrossRef]
29. Dragomir, S.S. *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*; Nova Science Publishers, Inc.: Hauppauge, NY, USA, 2005.
30. Mitrinović, D.S.; Pexcxarixcx, J.E.; Fink, A.M. *Classical and New Inequalities in Analysis*; Kluwer Academic Publishers: Amsterdam, The Netherlands, 1993.
31. Bombieri, E. A note on the large sieve. *Acta Arith.* **1971**, *18*, 401–404. [CrossRef]