# *Ostrowski and Trapezoid Type Inequalities for the Generalized k-g-Fractional Integrals of Functions with Bounded Variation*

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# Ostrowski and Trapezoid Type Inequalities for the Generalized $k$-$g$-Fractional Integrals of Functions with Bounded Variation

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**Abstract**
In this paper we establish some Ostrowski and trapezoid type inequalities for the $k$-$g$-fractional integrals of functions of bounded variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples for a general exponential fractional integral are also given.

**Keywords:** Functions of bounded variation, Generalized Riemann-Liouville fractional integrals, Hadamard fractional integrals, Ostrowski type inequalities, Trapezoid inequalities
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## 1. Introduction

Assume that the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function $k$ is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geq 1$, then $k$ is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *$k$-$g$-left-sided fractional integral* of $f$ by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b] \tag{1.1}$$

and the *$k$-$g$-right-sided fractional integral* of $f$ by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b). \tag{1.2}$$

If we take $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, where $\Gamma$ is the *Gamma function*, then

$$S_{k,g,a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x [g(x) - g(t)]^{\alpha-1} g'(t) f(t)\,dt$$
$$=: I^{\alpha}_{a+,g} f(x),\ a < x \le b$$

and

$$S_{k,g,b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b [g(t) - g(x)]^{\alpha-1} g'(t) f(t)\,dt \quad (1.3)$$
$$=: I^{\alpha}_{b-,g} f(x),\ a \le x < b,$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function $f$ with respect to another function $g$ on $[a,b]$ as defined in [1, p. 100]

For $g(t) = t$ in (1.3) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [1, p. 111]

$$H^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\,dt}{t},\ 0 \le a < x \le b$$

and

$$H^{\alpha}_{b-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\,dt}{t},\ 0 \le a < x < b.$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R^{\alpha}_{a+} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x \le b$$

and

$$R^{\alpha}_{b-} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x < b.$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the *"$\beta$-Exponential fractional integrals"*

$$E^{\alpha}_{a+,\beta} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x [\exp(\beta x) - \exp(\beta t)]^{\alpha-1} \exp(\beta t) f(t)\,dt,$$

for $a < x \le b$ and

$$E^{\alpha}_{b-,\beta} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b [\exp(\beta t) - \exp(\beta x)]^{\alpha-1} \exp(\beta t) f(t)\,dt,$$

for $a \le x < b$.

If we take $g(t) = t$ in (1.1) and (1.2), then we can consider the following *k-fractional integrals*

$$S_{k,a+} f(x) = \int_a^x k(x-t) f(t)\,dt,\ x \in (a,b] \quad (1.4)$$

and

$$S_{k,b-} f(x) = \int_x^b k(t-x) f(t)\,dt,\ x \in [a,b). \quad (1.5)$$

In [2], Raina studied a class of functions defined formally by

$$\mathscr{F}^{\sigma}_{\rho,\lambda}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k,\ |x| < R, \text{ with } R > 0 \quad (1.6)$$

for $\rho,\lambda>0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.6), Raina defined the following left-sided fractional integral operator

$$\mathscr{J}^{\sigma}_{\rho,\lambda,a+;w}f(x):=\int_a^x (x-t)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left(w(x-t)^{\rho}\right)f(t)\,dt,\ x>a \tag{1.7}$$

where $\rho,\lambda>0$, $w\in\mathbb{R}$ and $f$ is such that the integral on the right side exists.

In [3], the right-sided fractional operator was also introduced as

$$\mathscr{J}^{\sigma}_{\rho,\lambda,b-;w}f(x):=\int_x^b (t-x)^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}\left(w(t-x)^{\rho}\right)f(t)\,dt,\ x<b \tag{1.8}$$

where $\rho,\lambda>0$, $w\in\mathbb{R}$ and $f$ is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t)=t^{\lambda-1}\mathscr{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (1.7) and (1.8) from (1.4) and (1.5).

In [4], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathscr{T}^{\alpha}_{a+}f(x):=\frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\}f(t)\,dt,\ x>a \tag{1.9}$$

and

$$\mathscr{T}^{\alpha}_{b-}f(x):=\frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\}f(t)\,dt,\ x<b \tag{1.10}$$

where $\alpha\in(0,1)$.

We observe that for $k(t)=\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $t\in\mathbb{R}$ we re-obtain the definitions of (1.9) and (1.10) from (1.4) and (1.5).

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. We can define the more general exponential fractional integrals

$$\mathscr{T}^{\alpha}_{g,a+}f(x):=\frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\}g'(t)f(t)\,dt,\ x>a$$

and

$$\mathscr{T}^{\alpha}_{g,b-}f(x):=\frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\}g'(t)f(t)\,dt,\ x<b$$

where $\alpha\in(0,1)$.

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Assume that $\alpha>0$. We can also define the *logarithmic fractional integrals*

$$\mathscr{L}^{\alpha}_{g,a+}f(x):=\int_a^x (g(x)-g(t))^{\alpha-1}\ln(g(x)-g(t))\,g'(t)f(t)\,dt,$$

for $0<a<x\le b$ and

$$\mathscr{L}^{\alpha}_{g,b-}f(x):=\int_x^b (g(t)-g(x))^{\alpha-1}\ln(g(t)-g(x))\,g'(t)f(t)\,dt,$$

for $0<a\le x<b$, where $\alpha>0$. These are obtained from (1.4) and (1.5) for the kernel $k(t)=t^{\alpha-1}\ln t$, $t>0$.

For $\alpha=1$ we get

$$\mathscr{L}_{g,a+}f(x):=\int_a^x \ln(g(x)-g(t))\,g'(t)f(t)\,dt,\ 0<a<x\le b$$

and

$$\mathscr{L}_{g,b-}f(x):=\int_x^b \ln(g(t)-g(x))\,g'(t)f(t)\,dt,\ 0<a\le x<b.$$

For $g(t) = t$, we have the simple forms

$$\mathscr{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t)\,dt,\ 0 < a < x \le b,$$

$$\mathscr{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t)\,dt,\ 0 < a \le x < b,$$

$$\mathscr{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t)\,dt,\ 0 < a < x \le b$$

and

$$\mathscr{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t)\,dt,\ 0 < a \le x < b.$$

In the recent paper [5] we obtained the following Ostrowski and trapezoid type inequalities for the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function $f$ with respect to another function $g$ on $[a,b]$.

**Theorem 1.1.** *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Also let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Then we have*

$$\begin{aligned}
&\left| I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} \left[ (g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right] f(x) \right| \\
&\le \frac{1}{\Gamma(\alpha)} \left[ \int_a^x (g(t)-g(a))^{\alpha-1} g'(t) \bigvee_t^x (f)\,dt + \int_x^b (g(b)-g(t))^{\alpha-1} g'(t) \bigvee_x^t (f)\,dt \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \left[ (g(x)-g(a))^{\alpha} \bigvee_a^x (f) + (g(b)-g(x))^{\alpha} \bigvee_x^b (f) \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[ \frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \\ \left( (g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p} \right)^{1/p} \left( \left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q \right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left[ \frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \left( (g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right) \end{cases}
\end{aligned}$$

*and*

$$\begin{aligned}
&\left| I_{a+,g}^{\alpha} f(x) + I_{b-,g}^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)} \left[ (g(x)-g(a))^{\alpha} f(a) + (g(b)-g(x))^{\alpha} f(b) \right] \right| \\
&\le \frac{1}{\Gamma(\alpha)} \left[ \int_a^x (g(x)-g(t))^{\alpha-1} g'(t) \bigvee_a^t (f)\,dt + \int_x^b (g(t)-g(x))^{\alpha-1} g'(t) \bigvee_t^b (f)\,dt \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \left[ (g(x)-g(a))^{\alpha} \bigvee_a^x (f) + (g(b)-g(x))^{\alpha} \bigvee_x^b (f) \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[ \frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \\ \left( (g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p} \right)^{1/p} \left( \left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q \right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left[ \frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \left( (g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right) \end{cases}
\end{aligned}$$

*for any $x \in (a,b)$.*

For applications to the classical Riemann-Liouville fractional integrals, Hadamard fractional integrals and Harmonic fractional integrals see [5].

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [6]-[21], [22]-[32] and the references therein.

Motivated by the above results, in this paper we establish some Ostrowski and trapezoid type inequalities for the $k$-$g$-fractional integrals of functions of bounded variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples for a general exponential fractional integral are also given.

## 2. Some identities for the operator $S_{k,g,a+,b-}$

For $k$ and $g$ as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned} &S_{k,g,a+,b-}f(x) \\ &:= \frac{1}{2}\left[S_{k,g,a+}f(x) + S_{k,g,b-}f(x)\right] \\ &= \frac{1}{2}\left[\int_a^x k(g(x)-g(t))g'(t)f(t)\,dt + \int_x^b k(g(t)-g(x))g'(t)f(t)\,dt\right] \end{aligned}$$

for the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$ and $x\in(a,b)$.

The following two parameters representation for the operator $S_{k,g,a+,b-}$ holds:

**Lemma 2.1.** *With the above assumptions for $k$, $g$ and $f$ we have*

$$\begin{aligned} S_{k,g,a+,b-}f(x) = &\frac{1}{2}\left[\gamma K(g(b)-g(x)) + \lambda K(g(x)-g(a))\right] \\ &+ \frac{1}{2}\int_a^x k(g(x)-g(t))g'(t)[f(t)-\lambda]\,dt \\ &+ \frac{1}{2}\int_x^b k(g(t)-g(x))g'(t)[f(t)-\gamma]\,dt \end{aligned} \tag{2.1}$$

*for any $\lambda$, $\gamma\in\mathbb{C}$.*

*Proof.* We have, by taking the derivative over $t$ and using the chain rule, that

$$[K(g(x)-g(t))]' = K'(g(x)-g(t))(g(x)-g(t))' = -k(g(x)-g(t))g'(t)$$

for $t\in(a,x)$ and

$$[K(g(t)-g(x))]' = K'(g(t)-g(x))(g(t)-g(x))' = k(g(t)-g(x))g'(t)$$

for $t\in(x,b)$.

Therefore, for any $\lambda$, $\gamma\in\mathbb{C}$ we have

$$\begin{aligned} &\int_a^x k(g(x)-g(t))g'(t)[f(t)-\lambda]\,dt \\ &= \int_a^x k(g(x)-g(t))g'(t)f(t)\,dt - \lambda\int_a^x k(g(x)-g(t))g'(t)\,dt \\ &= S_{k,g,a+}f(x) + \lambda\int_a^x [K(g(x)-g(t))]'\,dt \\ &= S_{k,g,a+}f(x) + \lambda\left[K(g(x)-g(t))\right]\Big|_a^x = S_{k,g,a+}f(x) - \lambda K(g(x)-g(a)) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\int_x^b k(g(t)-g(x))g'(t)[f(t)-\gamma]\,dt \\ &= \int_x^b k(g(t)-g(x))g'(t)f(t)\,dt - \gamma\int_x^b k(g(t)-g(x))g'(t)\,dt \\ &= S_{k,g,b-}f(x) - \gamma\int_x^b [K(g(t)-g(x))]'\,dt \\ &= S_{k,g,b-}f(x) - \gamma\left[K(g(t)-g(x))\right]\Big|_x^b = S_{k,g,b-}f(x) - \gamma K(g(b)-g(x)) \end{aligned} \tag{2.3}$$

for $x \in (a,b)$.

If we add the equalities (2.2) and (2.3) and divide by 2 then we get the desired result (2.1). □

**Corollary 2.2.** *With the above assumptions for $k$, $g$ and $f$ we have the Ostrowski type identity*

$$\begin{aligned} S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x) \\ &+\frac{1}{2}\int_a^x k(g(x)-g(t))g'(t)\left[f(t)-f(x)\right]dt \\ &+\frac{1}{2}\int_x^b k(g(t)-g(x))g'(t)\left[f(t)-f(x)\right]dt \end{aligned} \tag{2.4}$$

*and the trapezoid type identity*

$$\begin{aligned} S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right] \\ &+\frac{1}{2}\int_a^x k(g(x)-g(t))g'(t)\left[f(t)-f(a)\right]dt \\ &+\frac{1}{2}\int_x^b k(g(t)-g(x))g'(t)\left[f(t)-f(b)\right]dt \end{aligned} \tag{2.5}$$

*for any $x \in (a,b)$.*

For $x = \frac{a+b}{2}$ we can consider

$$\begin{aligned} M_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f\left(\frac{a+b}{2}\right) \\ &= \frac{1}{2}\int_a^{\frac{a+b}{2}} k\left(g\left(\frac{a+b}{2}\right)-g(t)\right)g'(t)f(t)\,dt \\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^b k\left(g(t)-g\left(\frac{a+b}{2}\right)\right)g'(t)f(t)\,dt. \end{aligned}$$

By (2.4) we have the representation

$$\begin{aligned} &M_{k,g,a+,b-}f \\ &= \frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]f\left(\frac{a+b}{2}\right) \\ &+\frac{1}{2}\int_a^{\frac{a+b}{2}} k\left(g\left(\frac{a+b}{2}\right)-g(t)\right)g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt \\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^b k\left(g(t)-g\left(\frac{a+b}{2}\right)\right)g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt \end{aligned}$$

and (2.5) we have

$$\begin{aligned} &M_{k,g,a+,b-}f \\ &= \frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)f(b)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)f(a)\right] \\ &+\frac{1}{2}\int_a^{\frac{a+b}{2}} k\left(g\left(\frac{a+b}{2}\right)-g(t)\right)g'(t)\left[f(t)-f(a)\right]dt \\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^b k\left(g(t)-g\left(\frac{a+b}{2}\right)\right)g'(t)\left[f(t)-f(b)\right]dt. \end{aligned}$$

If $g$ is a function which maps an interval $I$ of the real line to the real numbers, and is both continuous and injective then we can define the *$g$-mean of two numbers $a$, $b \in I$* as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* $p$. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Using the $g$-mean of two numbers we can introduce

$$\begin{aligned}P_{k,g,a+,b-}f&:=S_{k,g,a+,b-}f\left(M_g(a,b)\right)\\&=\frac{1}{2}\int_a^{M_g(a,b)}k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)f(t)\,dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)f(t)\,dt.\end{aligned}$$

Using (2.4) and (2.5) we have the representations

$$\begin{aligned}&P_{k,g,a+,b-}f\\&=K\left(\frac{g(b)-g(a)}{2}\right)f\left(M_g(a,b)\right)\\&+\frac{1}{2}\int_a^{M_g(a,b)}k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)\left[f(t)-f\left(M_g(a,b)\right)\right]dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)\left[f(t)-f\left(M_g(a,b)\right)\right]dt\end{aligned}$$

and

$$\begin{aligned}&P_{k,g,a+,b-}f\\&=K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(b)+f(a)}{2}\\&+\frac{1}{2}\int_a^{M_g(a,b)}k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)\left[f(t)-f(a)\right]dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)\left[f(t)-f(b)\right]dt.\end{aligned}$$

## 3. Some identities for the dual operator $\breve{S}_{k,g,a+,b-}$

Observe that

$$S_{k,g,x+}f(b)=\int_x^b k\left(g(b)-g(t)\right)g'(t)f(t)\,dt,\ x\in[a,b)$$

and

$$S_{k,g,x-}f(a)=\int_a^x k\left(g(t)-g(a)\right)g'(t)f(t)\,dt,\ x\in(a,b].$$

Define also the mixed operator

$$\begin{aligned}&\breve{S}_{k,g,a+,b-}f(x)\\&:=\frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right]\\&=\frac{1}{2}\left[\int_x^b k\left(g(b)-g(t)\right)g'(t)f(t)\,dt+\int_a^x k\left(g(t)-g(a)\right)g'(t)f(t)\,dt\right]\end{aligned}$$

for any $x\in(a,b)$.

**Lemma 3.1.** *With the above assumptions for $k$, $g$ and $f$ we have*

$$\begin{aligned}\check{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[\lambda K(g(b)-g(x))+\gamma K(g(x)-g(a))\right] \\ &+\frac{1}{2}\int_a^x k(g(t)-g(a))g'(t)[f(t)-\gamma]\,dt \\ &+\frac{1}{2}\int_x^b k(g(b)-g(t))g'(t)[f(t)-\lambda]\,dt\end{aligned} \tag{3.1}$$

*for any $\lambda, \gamma \in \mathbb{C}$.*

*Proof.* We have, by taking the derivative over $t$ and using the chain rule, that

$$[K(g(b)-g(t))]' = K'(g(b)-g(t))(g(b)-g(t))' = -k(g(b)-g(t))g'(t)$$

for $t \in (x,b)$ and

$$[K(g(t)-g(a))]' = K'(g(t)-g(a))(g(t)-g(a))' = k(g(t)-g(a))g'(t)$$

for $t \in (a,x)$.

For any $\lambda, \gamma \in \mathbb{C}$ we have

$$\begin{aligned}&\int_x^b k(g(b)-g(t))g'(t)[f(t)-\lambda]\,dt \\ &= \int_x^b k(g(b)-g(t))g'(t)f(t)\,dt - \lambda\int_x^b k(g(b)-g(t))g'(t)\,dt \\ &= S_{k,g,x+}f(b) + \lambda\int_x^b [K(g(b)-g(t))]'\,dt \\ &= S_{k,g,x+}f(b) - \lambda K(g(b)-g(x))\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\int_a^x k(g(t)-g(a))g'(t)[f(t)-\gamma]\,dt \\ &= \int_a^x k(g(t)-g(a))g'(t)f(t)\,dt - \gamma\int_a^x k(g(t)-g(a))g'(t)\,dt \\ &= \int_a^x k(g(t)-g(a))g'(t)f(t)\,dt - \gamma\int_a^x [K(g(t)-g(a))]'\,dt \\ &= \int_a^x k(g(t)-g(a))g'(t)f(t)\,dt - \gamma K(g(x)-g(a))\end{aligned} \tag{3.3}$$

for $x \in (a,b)$.

If we add the equalities (3.2) and (3.3) and divide by 2 then we get the desired result (3.1). □

**Corollary 3.2.** *With the assumptions of Lemma 3.1 we have the Ostrowski type identity*

$$\begin{aligned}\check{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x) \\ &+\frac{1}{2}\int_a^x k(g(t)-g(a))g'(t)[f(t)-f(x)]\,dt \\ &+\frac{1}{2}\int_x^b k(g(b)-g(t))g'(t)[f(t)-f(x)]\,dt\end{aligned} \tag{3.4}$$

*and the trapezoid identity*

$$\begin{aligned}\check{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right] \\ &+\frac{1}{2}\int_a^x k(g(t)-g(a))g'(t)[f(t)-f(a)]\,dt \\ &+\frac{1}{2}\int_x^b k(g(b)-g(t))g'(t)[f(t)-f(b)]\,dt\end{aligned} \tag{3.5}$$

*for $x \in (a,b)$.*

For $x=\frac{a+b}{2}$ we can consider

$$\begin{aligned}\breve{M}_{k,g,a+,b-}f &:= \breve{S}_{k,g,a+,b-}f\left(\frac{a+b}{2}\right)\\ &= \frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt\\ &+\frac{1}{2}\int_{a}^{\frac{a+b}{2}} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt.\end{aligned}$$

Using the equalities (3.4) and (3.5), we have

$$\begin{aligned}&\breve{M}_{k,g,a+,b-}f\\ &=\frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right]f\left(\frac{a+b}{2}\right)\\ &+\frac{1}{2}\int_{a}^{\frac{a+b}{2}} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(\frac{a+b}{2}\right)\right]dt\\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(\frac{a+b}{2}\right)\right]dt\end{aligned}$$

and

$$\begin{aligned}&\breve{M}_{k,g,a+,b-}f\\ &=\frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)f\left(b\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)f\left(a\right)\right]\\ &+\frac{1}{2}\int_{a}^{\frac{a+b}{2}} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(a\right)\right]dt\\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(b\right)\right]dt.\end{aligned}$$

Using the $g$-mean of two numbers we can introduce

$$\begin{aligned}\breve{P}_{k,g,a+,b-}f &:= \breve{S}_{k,g,a+,b-}f\left(M_g\left(a,b\right)\right)\\ &= \frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt\\ &+\frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt.\end{aligned}$$

Using the equalities (3.4) and (3.5), we have

$$\begin{aligned}\breve{P}_{k,g,a+,b-}f &= K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)f\left(M_g\left(a,b\right)\right)\\ &+\frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(M_g\left(a,b\right)\right)\right]dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(M_g\left(a,b\right)\right)\right]dt\end{aligned}$$

and

$$\begin{aligned}\breve{P}_{k,g,a+,b-}f &= K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\frac{f\left(b\right)+f\left(a\right)}{2}\\ &+\frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(a\right)\right]dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(b\right)\right]dt.\end{aligned}$$

## 4. Trapezoid functional $T_{k,g,a+,b-}$

We can also introduce the functional

$$\begin{aligned}T_{k,g,a+,b-}f &:= \frac{1}{2}\left[S_{k,g,a+}f(b)+S_{k,g,b-}f(a)\right]\\ &= \frac{1}{2}\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\,f(t)\,dt.\end{aligned}$$

We have:

**Lemma 4.1.** *With the assumption of Lemma 2.1, we have*

$$\begin{aligned}T_{k,g,a+,b-}f &= K(g(b)-g(a))\,\delta \\ &\quad+\frac{1}{2}\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\,[f(t)-\delta]\,dt\end{aligned}\tag{4.1}$$

*for any $\delta\in\mathbb{C}$.*

*Proof.* Observe that

$$\begin{aligned}&\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\,dt\\ &=\int_a^b k(g(b)-g(t))\,g'(t)\,dt+\int_a^b k(g(t)-g(a))\,g'(t)\,dt\\ &=-\int_a^b [K(g(b)-g(t))]'\,dt+\int_a^b [K(g(t)-g(a))]'\,dt\\ &=-\,K(g(b)-g(t))\big|_a^b+K(g(t)-g(a))\big|_a^b\\ &=K(g(b)-g(a))+K(g(b)-g(a))=2K(g(b)-g(a)).\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{2}\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\,[f(t)-\delta]\,dt\\ &=\frac{1}{2}\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\,f(t)\,dt\\ &-\frac{1}{2}\delta\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\,dt\\ &=T_{k,g,a+,b-}f-\delta K(g(b)-g(a)),\end{aligned}$$

which proves the desired equality (4.1). ☐

**Corollary 4.2.** *With the assumptions of Lemma 4.1 we have the Ostrowski type identity*

$$\begin{aligned}&T_{k,g,a+,b-}f\\ &=K(g(b)-g(a))\,f(x)\\ &+\frac{1}{2}\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\,[f(t)-f(x)]\,dt\end{aligned}\tag{4.2}$$

*for any $x\in[a,b]$ and the trapezoid identity*

$$\begin{aligned}&T_{k,g,a+,b-}f\\ &=K(g(b)-g(a))\,\frac{f(a)+f(b)}{2}\\ &+\frac{1}{2}\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\left[f(t)-\frac{f(a)+f(b)}{2}\right]dt.\end{aligned}\tag{4.3}$$

We observe that for $x=\frac{a+b}{2}$ we obtain from (4.2) that

$$\begin{aligned}
&T_{k,g,a+,b-}f \\
&=K\left(g\left(b\right)-g\left(a\right)\right)f\left(\frac{a+b}{2}\right) \\
&+\frac{1}{2}\int_{a}^{b}\left[k\left(g\left(b\right)-g\left(t\right)\right)+k\left(g\left(t\right)-g\left(a\right)\right)\right]g'\left(t\right)\left[f\left(t\right)-f\left(\frac{a+b}{2}\right)\right]dt.
\end{aligned}$$

## 5. Inequalities for functions of bounded variation

We considered the cumulative function $K:[0,\infty)\to\mathbb{C}$ by

$$K\left(t\right):=\begin{cases}\int_{0}^{t}k\left(s\right)ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

We also define the function $\mathbf{K}:[0,\infty)\to[0,\infty)$ by

$$\mathbf{K}\left(t\right):=\begin{cases}\int_{0}^{t}\left|k\left(s\right)\right|ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

We observe that if $k$ takes nonnegative values on $(0,\infty)$, as it does in some of the examples in Introduction, then $\mathbf{K}\left(t\right)=K\left(t\right)$ for $t\in[0,\infty)$.

**Theorem 5.1.** *Assume that the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Then we have the Ostrowski type inequality*

$$\begin{aligned}
&\left|S_{k,g,a+,b-}f\left(x\right)-\frac{1}{2}\left[K\left(g\left(b\right)-g\left(x\right)\right)+K\left(g\left(x\right)-g\left(a\right)\right)\right]f\left(x\right)\right| \\
&\leq\frac{1}{2}\left[\int_{x}^{b}\left|k\left(g\left(t\right)-g\left(x\right)\right)\right|\bigvee_{x}^{t}\left(f\right)g'\left(t\right)dt+\int_{a}^{x}\left|k\left(g\left(x\right)-g\left(t\right)\right)\right|\bigvee_{t}^{x}\left(f\right)g'\left(t\right)dt\right] \\
&\leq\frac{1}{2}\left[\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)\bigvee_{x}^{b}\left(f\right)\ +\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\bigvee_{a}^{x}\left(f\right)\right] \\
&\leq\frac{1}{2}\begin{cases}\max\left\{\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right),\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right\}\bigvee_{a}^{b}\left(f\right); \\ \\ \left[\mathbf{K}^{p}\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}^{p}\left(g\left(x\right)-g\left(a\right)\right)\right]^{1/p}\left(\left(\bigvee_{a}^{x}\left(f\right)\right)^{q}+\left(\bigvee_{x}^{b}\left(f\right)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right]\left[\frac{1}{2}\bigvee_{a}^{b}\left(f\right)+\frac{1}{2}\left|\bigvee_{a}^{x}\left(f\right)-\bigvee_{x}^{b}\left(f\right)\right|\right]\end{cases}
\end{aligned} \tag{5.1}$$

*and the trapezoid type inequality*

$$
\begin{aligned}
&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right|\\
&\leq \frac{1}{2}\left[\int_a^x |k(g(x)-g(t))|\bigvee_a^t(f)g'(t)\,dt+\int_x^b |k(g(t)-g(x))|\bigvee_t^b(f)g'(t)\,dt\right]\\
&\leq \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right]\\
&\leq \frac{1}{2}\begin{cases}\max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b(f);\\[2mm]
[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))]^{1/p}\\
\times\left((\bigvee_a^x(f))^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm]
[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))]\\
\times\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}
\end{aligned}
\tag{5.2}
$$

*for any* $x\in(a,b)$.

*Proof.* Using the equality (2.4) we have

$$
\begin{aligned}
&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right|\\
&\leq \frac{1}{2}\left|\int_a^x k(g(x)-g(t))g'(t)[f(t)-f(x)]\,dt\right|\\
&+\frac{1}{2}\left|\int_x^b k(g(t)-g(x))g'(t)[f(t)-f(x)]\,dt\right|\\
&\leq \frac{1}{2}\int_a^x \left|k(g(x)-g(t))g'(t)[f(t)-f(x)]\right|dt\\
&+\frac{1}{2}\int_x^b \left|k(g(t)-g(x))g'(t)[f(t)-f(x)]\right|dt\\
&=\frac{1}{2}\int_a^x |k(g(x)-g(t))||f(x)-f(t)|g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(t)-g(x))||f(t)-f(x)|g'(t)\,dt\\
&=: B(x)
\end{aligned}
$$

for $x\in(a,b)$.

Since $f$ is of bounded variation, then

$$|f(x)-f(t)|\leq\bigvee_t^x(f)\leq\bigvee_a^x(f)\text{ for } a<t\leq x\leq b$$

and

$$|f(t)-f(x)|\leq\bigvee_x^t(f)\leq\bigvee_x^b(f)\text{ for } a\leq x\leq t<b.$$

Therefore

$$\begin{aligned}
B(x) &\leq \frac{1}{2}\int_a^x |k(g(x)-g(t))| \bigvee_t^x (f)\, g'(t)\,dt \\
&+\frac{1}{2}\int_x^b |k(g(t)-g(x))| \bigvee_x^t (f)\, g'(t)\,dt \\
&\leq \frac{1}{2}\bigvee_a^x (f)\int_a^x |k(g(x)-g(t))|\, g'(t)\,dt \\
&+\frac{1}{2}\bigvee_x^b (f)\int_x^b |k(g(t)-g(x))|\, g'(t)\,dt \\
&=: C(x)
\end{aligned}$$

for $x \in (a,b)$.

We have, by taking the derivative over $t$ and using the chain rule, that

$$[\mathbf{K}(g(x)-g(t))]' = \mathbf{K}'(g(x)-g(t))(g(x)-g(t))' = -|k(g(x)-g(t))|\, g'(t)$$

for $t \in (a,x)$ and

$$[\mathbf{K}(g(t)-g(x))]' = \mathbf{K}'(g(t)-g(x))(g(t)-g(x))' = |k(g(t)-g(x))|\, g'(t)$$

for $t \in (x,b)$.

Then

$$\int_a^x |k(g(x)-g(t))|\, g'(t)\,dt = -\int_a^x [\mathbf{K}(g(x)-g(t))]'\,dt = \mathbf{K}(g(x)-g(a))$$

and

$$\int_x^b |k(g(t)-g(x))|\, g'(t)\,dt = \int_x^b [\mathbf{K}(g(t)-g(x))]'\,dt = \mathbf{K}(g(b)-g(x))$$

giving that

$$C(x) = \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b (f) + \mathbf{K}(g(x)-g(a))\bigvee_a^x (f)\right],$$

for $x \in (a,b)$, which proves the first and the second inequality in (5.1).

The last part of (4.1 is obvious by making use of the elementary Hölder type inequalities for positive real numbers $c$, $d$, $m$, $n \geq 0$

$$mc+nd \leq \begin{cases} \max\{m,n\}(c+d); \\ (m^p+n^p)^{1/p}(c^q+d^q)^{1/q} \text{ with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

Further, by the identity (2.5) we have, as above,

$$\begin{aligned}
&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right|\\
&\le \frac{1}{2}\int_a^x |k(g(x)-g(t))|\,|f(t)-f(a)|\,g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(t)-g(x))|\,|f(t)-f(b)|\,g'(t)\,dt\\
&\le \frac{1}{2}\int_a^x |k(g(x)-g(t))|\bigvee_a^t(f)\,g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(t)-g(x))|\bigvee_t^b(f)\,g'(t)\,dt\\
&\le \frac{1}{2}\bigvee_a^x(f)\int_a^x |k(g(x)-g(t))|\,g'(t)\,dt\\
&+\frac{1}{2}\bigvee_x^b(f)\int_x^b |k(g(t)-g(x))|\,g'(t)\,dt\\
&=\frac{1}{2}\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\frac{1}{2}\mathbf{K}(g(b)-g(x))\bigvee_x^b(f),
\end{aligned}$$

which proves (5.2). □

The following particular case for the functional

$$\begin{aligned}
P_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f(M_g(a,b))\\
&=\frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(b)+g(a)}{2}-g(t)\right)g'(t)\,f(t)\,dt\\
&+\frac{1}{2}\int_{M_g(a,b)}^b k\left(g(t)-\frac{g(b)+g(a)}{2}\right)g'(t)\,f(t)\,dt.
\end{aligned}$$

is of interest:

**Corollary 5.2.** *With the assumptions of Theorem 5.1 we have*

$$\begin{aligned}
\left|P_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)f(M_g(a,b))\right| &\le \frac{1}{2}\int_{M_g(a,b)}^b\left|k\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right|\bigvee_{M_g(a,b)}^t(f)\,g'(t)\,dt\\
&+\frac{1}{2}\int_a^{M_g(a,b)}\left|k\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right|\bigvee_t^{M_g(a,b)}(f)\,g'(t)\,dt\\
&\le \frac{1}{2}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_b^b(f) \qquad (5.3)
\end{aligned}$$

*and*

$$\begin{aligned}
\left|P_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(b)+f(a)}{2}\right| &\le \frac{1}{2}\int_a^{M_g(a,b)}\left|k\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right|\bigvee_a^t(f)\,g'(t)\,dt\\
&+\frac{1}{2}\int_{M_g(a,b)}^b\left|k\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right|\bigvee_t^b(f)\,g'(t)\,dt\\
&\le \frac{1}{2}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_b^b(f). \qquad (5.4)
\end{aligned}$$

We have:

**Theorem 5.3.** *With the assumptions of Theorem 5.1 we have the Ostrowski type inequality*

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))| \bigvee_t^x (f)\, g'(t)\,dt+\frac{1}{2}\int_x^b |k(g(b)-g(t))| \bigvee_x^t (f)\, g'(t)\,dt \\
&\leq \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b (f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x (f)\right] \\
&\leq \frac{1}{2}\begin{cases}
\max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b (f); \\[1ex]
\left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x (f)\right)^q+\left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[1ex]
\left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b (f)+\frac{1}{2}\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right]
\end{cases} \qquad (5.5)
\end{aligned}$$

*and the trapezoid inequality*

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right| \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))| \bigvee_a^t (f)\, g'(t)\,dt+\frac{1}{2}\int_x^b |k(g(b)-g(t))| \bigvee_t^b (f)\, g'(t)\,dt \\
&\leq \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b (f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x (f)\right] \\
&\leq \frac{1}{2}\begin{cases}
\max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b (f); \\[1ex]
\left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x (f)\right)^q+\left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[1ex]
\left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b (f)+\frac{1}{2}\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right]
\end{cases} \qquad (5.6)
\end{aligned}$$

*for any $x\in(a,b)$.*

*Proof.* Using the identity (3.4) we have

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right|\\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\,|f(t)-f(x)|\,g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\,|f(t)-f(x)|\,g'(t)\,dt\\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\bigvee_t^x(f)\,g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\bigvee_x^t(f)\,g'(t)\,dt\\
&\leq \frac{1}{2}\bigvee_a^x(f)\int_a^x |k(g(t)-g(a))|\,g'(t)\,dt\\
&+\frac{1}{2}\bigvee_x^b(f)\int_x^b |k(g(b)-g(t))|\,g'(t)\,dt\\
&=\frac{1}{2}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right],
\end{aligned}$$

for any $x\in(a,b)$, which proves (5.5).

By the identity (3.5) we have

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right|\\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\,|f(t)-f(a)|\,g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\,|f(b)-f(t)|\,g'(t)\,dt\\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\bigvee_a^t(f)\,g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\bigvee_t^b(f)\,g'(t)\,dt\\
&\leq \frac{1}{2}\bigvee_a^x(f)\int_a^x |k(g(t)-g(a))|\,g'(t)\,dt\\
&+\frac{1}{2}\bigvee_x^b(f)\int_x^b |k(g(b)-g(t))|\,g'(t)\,dt\\
&=\frac{1}{2}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right]
\end{aligned}$$

for any $x\in(a,b)$, which proves (5.6). □

Also, we have the particular inequalities for

$$\begin{aligned}
\breve{P}_{k,g,a+,b-}f &:= \breve{S}_{k,g,a+,b-}f(M_g(a,b))\\
&=\frac{1}{2}\int_{M_g(a,b)}^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt\\
&+\frac{1}{2}\int_a^{M_g(a,b)} k(g(t)-g(a))\,g'(t)\,f(t)\,dt.
\end{aligned}$$

**Corollary 5.4.** *With the assumptions of Theorem 5.1 we have*

$$\begin{aligned}\left|\check{P}_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(b)+f(a)}{2}\right| &\leq \frac{1}{2}\int_a^{M_g(a,b)}|k(g(t)-g(a))|\bigvee_t^{M_g(a,b)}(f)g'(t)\,dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^b|k(g(b)-g(t))|\bigvee_{M_g(a,b)}^t(f)g'(t)\,dt\\ &\leq \frac{1}{2}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_b^b(f)\end{aligned}$$

*and*

$$\begin{aligned}\left|\check{P}_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(b)+f(a)}{2}\right| &\leq \frac{1}{2}\int_a^{M_g(a,b)}|k(g(t)-g(a))|\bigvee_a^t(f)g'(t)\,dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^b|k(g(b)-g(t))|\bigvee_t^b(f)g'(t)\,dt\\ &\leq \frac{1}{2}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_b^b(f).\end{aligned}$$

Finally, we have the following result for the trapezoid functional

$$\begin{aligned}T_{k,g,a+,b-}f &:= \frac{1}{2}\left[S_{k,g,a+}f(b)+S_{k,g,b-}f(a)\right]\\ &= \frac{1}{2}\int_a^b[k(g(b)-g(t))+k(g(t)-g(a))]g'(t)f(t)\,dt.\end{aligned}$$

**Theorem 5.5.** *With the assumptions of Theorem 5.1 we have the trapezoid type inequality*

$$\left|T_{k,g,a+,b-}f-K(g(b)-g(a))\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{2}\mathbf{K}(g(b)-g(a))\bigvee_a^b(f). \tag{5.7}$$

*Proof.* From the identity (4.3) we have

$$\begin{aligned}&\left|T_{k,g,a+,b-}f-K(g(b)-g(a))\frac{f(a)+f(b)}{2}\right|\\ &\leq\frac{1}{2}\int_a^b|k(g(b)-g(t))+k(g(t)-g(a))|\left|f(t)-\frac{f(a)+f(b)}{2}\right|g'(t)\,dt\\ &\leq\frac{1}{2}\int_a^b[|k(g(b)-g(t))|+|k(g(t)-g(a))|]\left|f(t)-\frac{f(a)+f(b)}{2}\right|g'(t)\,dt\\ &=:D.\end{aligned}$$

Since $f:[a,b]\to\mathbb{C}$ is of bounded variation, then for any $t\in[a,b]$ we have

$$\begin{aligned}\left|f(t)-\frac{f(a)+f(b)}{2}\right| &= \left|\frac{f(t)-f(a)+f(t)-f(b)}{2}\right|\\ &\leq\frac{1}{2}[|f(t)-f(a)|+|f(b)-f(t)|]\leq\frac{1}{2}\bigvee_a^b(f).\end{aligned}$$

Therefore

$$\begin{aligned}D &\leq \frac{1}{4}\bigvee_a^b(f)\int_a^b[|k(g(b)-g(t))|+|k(g(t)-g(a))|]g'(t)\,dt\\ &=\frac{1}{4}\bigvee_a^b(f)[\mathbf{K}(g(b)-g(a))+\mathbf{K}(g(b)-g(a))]=\frac{1}{2}\mathbf{K}(g(b)-g(a))\bigvee_a^b(f),\end{aligned}$$

which proves the desired result (5.7). □

## 6. Example for an exponential kernel

The above inequalities may be written for all the particular fractional integrals introduced in the introduction.

If we take, for instance $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, where $\Gamma$ is the *Gamma function*, then we recapture the results for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function $f$ with respect to another function $g$ on $[a,b]$ as outlined in [5].

For $\alpha$, $\beta \in \mathbb{R}$ we consider the kernel $k(t) := \exp[(\alpha+\beta i)t]$, $t \in \mathbb{R}$. We have

$$K(t) = \frac{\exp[(\alpha+\beta i)t] - 1}{(\alpha+\beta i)}, \text{ if } t \in \mathbb{R}$$

for $\alpha$, $\beta \neq 0$.

Also, we have

$$|k(s)| := |\exp[(\alpha+\beta i)s]| = \exp(\alpha s) \text{ for } s \in \mathbb{R}$$

and

$$\mathbf{K}(t) = \int_0^t \exp(\alpha s)\, ds = \frac{\exp(\alpha t) - 1}{\alpha} \text{ if } 0 < t,$$

for $\alpha \neq 0$.

Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. We have

$$\begin{aligned}\mathscr{E}^{\alpha+\beta i}_{g,a+,b-} f(x) &= \frac{1}{2}\int_a^x \exp[(\alpha+\beta i)(g(x)-g(t))]\, g'(t) f(t)\, dt \\ &+ \frac{1}{2}\int_x^b \exp[(\alpha+\beta i)(g(t)-g(x))]\, g'(t) f(t)\, dt\end{aligned}$$

for $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on $(a,b)$, having a continuous derivative $h'$ on $(a,b)$, then we can consider the following operator as well

$$\begin{aligned}&\kappa^{\alpha+\beta i}_{h,a+,b-} f(x) \\ &:= \mathscr{E}^{\alpha+\beta i}_{\ln h,a+,b-} f(x) \\ &= \frac{1}{2}\left[\int_a^x \left(\frac{h(x)}{h(t)}\right)^{\alpha+\beta i} \frac{h'(t)}{h(t)} f(t)\, dt + \int_x^b \left(\frac{h(t)}{h(x)}\right)^{\alpha+\beta i} \frac{h'(t)}{h(t)} f(t)\, dt\right],\end{aligned}$$

for $x \in (a,b)$.

By using the inequality (5.1) we have for $x \in (a,b)$ that

$$
\begin{aligned}
&\left|\mathscr{E}_{g,a+,b-}^{\alpha+\beta i} f(x)\right. \\
&\left.-\frac{1}{2}\left[\frac{\exp[(\alpha+\beta i)(g(b)-g(x))]-1+\exp[(\alpha+\beta i)(g(x)-g(a))]-1}{(\alpha+\beta i)}\right] f(x)\right| \\
&\leq \frac{1}{2}\left[\int_x^b \exp(\alpha(g(t)-g(x))) g'(t) \bigvee_x^t (f)\, dt+\int_a^x \exp(\alpha(g(x)-g(t))) g'(t) \bigvee_t^x (f)\, dt\right] \\
&\leq \frac{1}{2}\left[\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha} \bigvee_x^b (f)\ +\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha} \bigvee_a^x (f)\right] \\
&\qquad\qquad \leq \frac{1}{2}\begin{cases}
\max\left\{\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}, \frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right\} \bigvee_a^b (f); \\
\\
\left[\left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p+\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p\right]^{1/p}\left(\left(\bigvee_a^x (f)\right)^q+\left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\left[\frac{\exp(\alpha(g(b)-g(x)))-1+\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right]\left[\frac{1}{2}\bigvee_a^b (f)+\frac{1}{2}\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right]
\end{cases}
\end{aligned}
$$

for $\alpha,\ \beta \in \mathbb{R}$ with $\alpha \neq 0$.

By using the inequality (5.2) we also have for $x \in (a,b)$ that

$$
\begin{aligned}
&\left|\mathscr{E}_{g,a+,b-}^{\alpha+\beta i} f(x)-\frac{1}{2}\left[\frac{(\exp[(\alpha+\beta i)(g(b)-g(x))]-1) f(b)+(\exp[(\alpha+\beta i)(g(x)-g(a))]-1) f(a)}{(\alpha+\beta i)}\right]\right| \\
&\leq \frac{1}{2}\left[\int_a^x \exp(\alpha(g(t)-g(x))) g'(t) \bigvee_a^t (f)\, dt+\int_x^b \exp(\alpha(g(x)-g(t))) g'(t) \bigvee_t^b (f)\, dt\right] \\
&\leq \frac{1}{2}\left[\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha} \bigvee_x^b (f)\ +\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha} \bigvee_a^x (f)\right] \\
&\leq \frac{1}{2}\begin{cases}
\max\left\{\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}, \frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right\} \bigvee_a^b (f); \\
\\
\left[\left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p+\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p\right]^{1/p} \\
\times\left(\left(\bigvee_a^x (f)\right)^q+\left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\left[\frac{\exp(\alpha(g(b)-g(x)))-1+\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right] \\
\times\left[\frac{1}{2}\bigvee_a^b (f)+\frac{1}{2}\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right]
\end{cases}
\end{aligned}
$$

for $\alpha,\ \beta \in \mathbb{R}$ with $\alpha \neq 0$.

If we denote

$$
\begin{aligned}
\overline{\mathscr{E}}_{g,a+,b-}^{\alpha+\beta i} f &:= \mathscr{E}_{g,a+,b-}^{\alpha+\beta i} f(M_g(a,b)) \\
&= \frac{1}{2}\int_a^x \exp\left[(\alpha+\beta i)\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right] g'(t) f(t)\, dt \\
&+\frac{1}{2}\int_x^b \exp\left[(\alpha+\beta i)\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right] g'(t) f(t)\, dt
\end{aligned}
$$

then by (5.3) and (5.4) we have the simpler results

$$\begin{aligned}\left|\overline{\mathscr{E}}^{\alpha+\beta i}_{g,a+,b-}f-\frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}f\left(M_g(a,b)\right)\right| &\le \frac{1}{2}\int_{M_g(a,b)}^{b}\exp\left(\alpha\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right)g'(t)\bigvee_{M_g(a,b)}^{t}(f)\,dt\\ &+\frac{1}{2}\int_{a}^{M_g(a,b)}\exp\left(\alpha\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right)g'(t)\bigvee_{t}^{M_g(a,b)}(f)\,dt\\ &\le \frac{1}{2}\frac{\exp\left(\alpha\left(\frac{g(b)-g(a)}{2}\right)\right)-1}{\alpha}\bigvee_{b}^{b}(f) \end{aligned}\tag{6.1}$$

and

$$\begin{aligned}\left|\overline{\mathscr{E}}^{\alpha+\beta i}_{g,a+,b-}f-\frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}\frac{f(b)+f(a)}{2}\right| &\le \frac{1}{2}\int_{a}^{M_g(a,b)}\exp\left(\alpha\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right)g'(t)\bigvee_{a}^{t}(f)\,dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^{b}\exp\left(\alpha\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right)g'(t)\bigvee_{t}^{b}(f)\,dt\\ &\le \frac{1}{2}\frac{\exp\left(\alpha\left(\frac{g(b)-g(a)}{2}\right)\right)-1}{\alpha}\bigvee_{b}^{b}(f). \end{aligned}\tag{6.2}$$

In particular, if we take in (6.1) and (6.2) $g=\ln t$, $t\in[a,b]\subset(0,\infty)$, then by using the notation $G(\gamma,\delta):=\sqrt{\gamma\delta}$ for the *geometric mean* of the positive real numbers $\gamma$, $\delta>0$ we have

$$\begin{aligned}\left|\bar{\kappa}^{\alpha+\beta i}_{a+,b-}f-\frac{\left(\frac{b}{a}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}f(G(a,b))\right| &\le \frac{1}{2}\int_{G(a,b)}^{b}\left(\frac{t}{G(a,b)}\right)^{\alpha}\frac{1}{t}\bigvee_{G(a,b)}^{t}(f)\,dt+\frac{1}{2}\int_{a}^{G(a,b)}\left(\frac{G(a,b)}{t}\right)^{\alpha}\frac{1}{t}\bigvee_{t}^{G(a,b)}(f)\,dt\\ &\le \frac{1}{2}\frac{\left(\frac{b}{a}\right)^{\alpha}-1}{\alpha}\bigvee_{b}^{b}(f)\end{aligned}$$

and

$$\begin{aligned}\left|\bar{\kappa}^{\alpha+\beta i}_{a+,b-}f-\frac{\left(\frac{b}{a}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\frac{f(b)+f(a)}{2}\right| &\le \frac{1}{2}\int_{G(a,b)}^{b}\left(\frac{G(a,b)}{t}\right)^{\alpha}\frac{1}{t}\bigvee_{t}^{b}(f)\,dt+\frac{1}{2}\int_{a}^{G(a,b)}\left(\frac{t}{G(a,b)}\right)^{\alpha}\frac{1}{t}\bigvee_{a}^{t}(f)\,dt\\ &\le \frac{1}{2}\frac{\left(\frac{b}{a}\right)^{\alpha}-1}{\alpha}\bigvee_{b}^{b}(f),\end{aligned}$$

where

$$\bar{\kappa}^{\alpha+\beta i}_{a+,b-}f:=\frac{1}{2}\int_{G(a,b)}^{b}\left(\frac{t}{G(a,b)}\right)^{\alpha+\beta i}\frac{1}{t}f(t)\,dt+\frac{1}{2}\int_{a}^{G(a,b)}\left(\frac{G(a,b)}{t}\right)^{\alpha+\beta i}\frac{1}{t}f(t)\,dt.$$

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