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Tensorial and Hadamard Product Inequalities for Synchronous Functions

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Abstract
Let H be a Hilbert space. In this paper we show among others that, if f, g are synchronous and continuous on I and A, B are selfadjoint with spectra $Sp(A)$, $Sp(B) \subset I$, then

$$(f(A)g(A)) \otimes 1 + 1 \otimes (f(B)g(B)) \geq f(A) \otimes g(B) + g(A) \otimes f(B)$$

and the inequality for Hadamard product

$$(f(A)g(A) + f(B)g(B)) \circ 1 \geq f(A) \circ g(B) + f(B) \circ g(A).$$

Let either $p,q \in (0,\infty)$ or $p,q \in (-\infty,0)$. If A, $B > 0$, then

$$A^{p+q} \otimes 1 + 1 \otimes B^{p+q} \geq A^p \otimes B^q + A^q \otimes B^p,$$

and

$$\left(A^{p+q} + B^{p+q}\right) \circ 1 \geq A^p \circ B^q + A^q \circ B^p.$$

Keywords: Convex functions, Hadamard Product, Selfadjoint operators, Tensorial product
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1. Introduction

Let $I_1, \ldots, I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times \ldots \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, \ldots, A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, \ldots, H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, \ldots, k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, \ldots, k$; by following [1], we define

$$f(A_1, \ldots, A_k) := \int_{I_1} \ldots \int_{I_k} f(\lambda_1, \ldots, \lambda_k)\, dE_1(\lambda_1) \otimes \ldots \otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes \ldots \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [1] extends the definition of Korányi [2] for functions of two variables and have the property that

$$f(A_1, \ldots, A_k) = f_1(A_1) \otimes \ldots \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, \ldots, t_k) = f_1(t_1) \ldots f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [3, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.3}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2}$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [4] obtained the following *Callebaut type inequalities* for tensorial product

$$(A \# B) \otimes (A \# B) \leq \frac{1}{2}\left[(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B)\right] \leq \frac{1}{2}(A \otimes B + B \otimes A) \tag{1.4}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\left\langle (A \circ B) e_j, e_j \right\rangle = \left\langle A e_j, e_j \right\rangle \left\langle B e_j, e_j \right\rangle$$

for all $j \in \mathbb{N}$, where $\left\{e_j\right\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H. It is known that, see [5], we have the representation

$$A \circ B = \mathscr{U}^* (A \otimes B) \mathscr{U} \tag{1.5}$$

where $\mathscr{U} : H \to H \otimes H$ is the isometry defined by $\mathscr{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative and operator concave (sub-multiplicative and operator convex)* on $[0, \infty)$, then also [3, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.6}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2}\right) \circ 1 \text{ for } A,\ B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [6] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [7] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [8] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

For other inequalities concerning tensorial product, see [9] and [10].

Motivated by the above results, in this paper we show among others that if f, g are synchronous and continuous on I and A, B are selfadjoint with spectra $Sp(A)$, $Sp(B) \subset I$, then

$$(f(A)g(A)) \otimes 1 + 1 \otimes (f(B)g(B)) \geq f(A) \otimes g(B) + g(A) \otimes f(B)$$

and the inequality for Hadamard product

$$(f(A)g(A) + f(B)g(B)) \circ 1 \geq f(A) \circ g(B) + f(B) \circ g(A).$$

Let either $p, q \in (0, \infty)$ or $p, q \in (-\infty, 0)$. If A, $B > 0$, then

$$A^{p+q} \otimes 1 + 1 \otimes B^{p+q} \geq A^p \otimes B^q + A^q \otimes B^p,$$

and

$$\left(A^{p+q} + B^{p+q}\right) \circ 1 \geq A^p \circ B^q + A^q \circ B^p.$$

2. Main Results

We start with the following main result:

Theorem 2.1. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If A, B are selfadjoint with spectra $Sp(A)$, $Sp(B) \subset I$, then*

$$[h(A)f(A)g(A)] \otimes k(B) + h(A) \otimes [k(B)f(B)g(B)] \geq [h(A)f(A)] \otimes [k(B)g(B)] + [h(A)g(A)] \otimes [k(B)f(B)] \tag{2.1}$$

or, equivalently

$$(h(A) \otimes k(B))\,[(f(A)g(A)) \otimes 1 + 1 \otimes (f(B)g(B))] \geq (h(A) \otimes k(B))\,[f(A) \otimes g(B) + g(A) \otimes f(B)]. \tag{2.2}$$

If f, g are asynchronous on I, then the inequality reverses in (2.1) and (2.2).

Proof. Assume that f and g are synchronous on I, then

$$f(t)g(t) + f(s)g(s) \geq f(t)g(s) + f(s)g(t)$$

for all $t, s \in I$. We multiply this inequality by $h(t)k(s) \geq 0$ to get

$$f(t)g(t)h(t)k(s) + h(t)f(s)g(s)k(s) \geq f(t)h(t)g(s)k(s) + f(s)k(s)g(t)h(t)$$

for all $t,s\in I$. If we take the double integral, then we get

$$\begin{aligned}&\int_I\int_I [f(t)g(t)h(t)k(s)+h(t)f(s)g(s)k(s)]\,dE(t)\otimes dF(s)\\&\geq \int_I\int_I [f(t)h(t)g(s)k(s)+f(s)k(s)g(t)h(t)]\,dE(t)\otimes dF(s).\end{aligned} \tag{2.3}$$

Observe that

$$\begin{aligned}\int_I\int_I [f(t)g(t)h(t)k(s)+h(t)f(s)g(s)k(s)]\,dE(t)\otimes dF(s) &= \int_I\int_I f(t)g(t)h(t)k(s)\,dE(t)\otimes dF(s)\\&\quad+\int_I\int_I h(t)f(s)g(s)k(s)\,dE(t)\otimes dF(s)\\&=[h(A)f(A)g(A)]\otimes k(B)+h(A)\otimes[k(B)f(B)g(B)]\end{aligned}$$

and

$$\begin{aligned}\int_I\int_I [f(t)h(t)g(s)k(s)+f(s)k(s)g(t)h(t)]\,dE(t)\otimes dF(s) &= \int_I\int_I f(t)h(t)g(s)k(s)\,dE(t)\otimes dF(s)\\&\quad+\int_I\int_I g(t)h(t)f(s)k(s)\,dE(t)\otimes dF(s)\\&=[h(A)f(A)]\otimes[k(B)g(B)]+[h(A)g(A)]\otimes[k(B)f(B)].\end{aligned}$$

By utilizing (2.3) we derive (2.2). Now, by making use of the tensorial property

$$(XU)\otimes(YV)=(X\otimes Y)(U\otimes V),$$

for any $X,U,Y,V\in B(H)$, we obtain

$$\begin{aligned}&[h(A)f(A)g(A)]\otimes k(B)+h(A)\otimes[k(B)f(B)g(B)]\\&=(h(A)\otimes k(B))[(f(A)g(A))\otimes 1]+(h(A)\otimes k(B))[1\otimes(f(B)g(B))]\\&=(h(A)\otimes k(B))[(f(A)g(A))\otimes 1+1\otimes(f(B)g(B))]\end{aligned}$$

and

$$\begin{aligned}&[h(A)f(A)]\otimes[k(B)g(B)]+[h(A)g(A)]\otimes[k(B)f(B)]\\&=(h(A)\otimes k(B))(f(A)\otimes g(B))+(h(A)\otimes k(B))(g(A)\otimes f(B))\\&=(h(A)\otimes k(B))[f(A)\otimes g(B)+g(A)\otimes f(B)],\end{aligned}$$

which proves (2.2). $\square$

Remark 2.2. *With the assumptions of Theorem 2.1 and if we take $k=h$, then we get*

$$[h(A)f(A)g(A)]\otimes h(B)+h(A)\otimes[h(B)f(B)g(B)]\geq[h(A)f(A)]\otimes[h(B)g(B)]+[h(A)g(A)]\otimes[h(B)f(B)], \tag{2.4}$$

where f,g are synchronous and continuous on I and h is nonnegative and continuous on the same interval. Moreover, if we take $h\equiv 1$ in (2.4), then we get

$$(f(A)g(A))\otimes 1+1\otimes(f(B)g(B))\geq f(A)\otimes g(B)+g(A)\otimes f(B), \tag{2.5}$$

where f,g are synchronous and continuous on I

Corollary 2.3. *Assume that f,g are synchronous and continuous on I and h,k nonnegative and continuous on the same interval. If A,B are selfadjoint with spectra $Sp(A), Sp(B)\subset I$, then*

$$k(B)\circ[h(A)f(A)g(A)]+h(A)\circ[k(B)f(B)g(B)]\geq[h(A)f(A)]\circ[k(B)g(B)]+[k(B)f(B)]\circ[h(A)g(A)]. \tag{2.6}$$

If f,g are asynchronous on I, then the inequality reverses in (2.6). In particular, we have

$$h(B)\circ[h(A)f(A)g(A)]+h(A)\circ[h(B)f(B)g(B)]\geq[h(A)f(A)]\circ[h(B)g(B)]+[h(B)f(B)]\circ[h(A)g(A)] \tag{2.7}$$

and

$$(f(A)g(A)+(f(B)g(B)))\circ 1\geq f(A)\circ g(B)+f(B)\circ g(A). \tag{2.8}$$

Proof. If we take $\mathscr{U}^*$ to the left and $\mathscr{U}$ to the right in the inequality (2.1), we get

$$\begin{aligned}\mathscr{U}^*\left([h(A)f(A)g(A)]\otimes k(B)\right)\mathscr{U}+\mathscr{U}^*\left(h(A)\otimes[k(B)f(B)g(B)]\right)\mathscr{U}\geq&\mathscr{U}^*\left([h(A)f(A)]\otimes[k(B)g(B)]\right)\mathscr{U}\\&+\mathscr{U}^*\left([h(A)g(A)]\otimes[k(B)f(B)]\right)\mathscr{U}\end{aligned}$$

which is equivalent to (2.6). □

Corollary 2.4. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If A_j, B_j are selfadjoint with spectra $Sp(A_j)$, $Sp(B_j)\subset I$ and $p_j,q_j\geq 0$, $j\in\{1,\ldots,n\}$, then*

$$\begin{aligned}&\left(\sum_{j=1}^n p_jh(A_j)f(A_j)g(A_j)\right)\otimes\left(\sum_{i=1}^n q_ik(B_i)\right)+\left(\sum_{j=1}^n p_jh(A_j)\right)\otimes\left(\sum_{i=1}^n q_ik(B_i)f(B_i)g(B_i)\right)\\&\geq\left(\sum_{j=1}^n p_jh(A_j)f(A_j)\right)\otimes\left(\sum_{i=1}^n q_ik(B_i)g(B_i)\right)+\left(\sum_{j=1}^n p_jh(A_j)g(A_j)\right)\otimes\left(\sum_{i=1}^n q_ik(B_i)f(B_i)\right).\end{aligned}\tag{2.9}$$

In particular,

$$\begin{aligned}&\left(\sum_{j=1}^n p_jh(A_j)f(A_j)g(A_j)\right)\otimes\left(\sum_{i=1}^n q_ih(B_i)\right)+\left(\sum_{j=1}^n p_jh(A_j)\right)\otimes\left(\sum_{i=1}^n q_ih(B_i)f(B_i)g(B_i)\right)\\&\geq\left(\sum_{j=1}^n p_jh(A_j)f(A_j)\right)\otimes\left(\sum_{i=1}^n q_ih(B_i)g(B_i)\right)+\left(\sum_{j=1}^n p_jh(A_j)g(A_j)\right)\otimes\left(\sum_{i=1}^n q_ih(B_i)f(B_i)\right)\end{aligned}\tag{2.10}$$

and, if $\sum_{j=1}^n p_j=\sum_{j=1}^n q_j=1$, then

$$\begin{aligned}\left(\sum_{j=1}^n p_jf(A_j)g(A_j)\right)\otimes 1+1\otimes\left(\sum_{i=1}^n q_if(B_i)g(B_i)\right)\geq&\left(\sum_{j=1}^n p_jf(A_j)\right)\otimes\left(\sum_{i=1}^n q_ig(B_i)\right)\\&+\left(\sum_{j=1}^n p_jg(A_j)\right)\otimes\left(\sum_{i=1}^n q_if(B_i)\right).\end{aligned}\tag{2.11}$$

Proof. We have from (2.1) that

$$\begin{aligned}[h(A_j)f(A_j)g(A_j)]\otimes k(B_i)+h(A_j)\otimes[k(B_i)f(B_i)g(B_i)]\geq&[h(A_j)f(A_j)]\otimes[k(B_i)g(B_i)]\\&+[h(A_j)g(A_j)]\otimes[k(B_i)f(B_i)]\end{aligned}$$

for all $i,j\in\{1,\ldots,n\}$. If we multiply by $p_jq_i\geq 0$ and sum over $j,i\in\{1,\ldots,n\}$, then we get

$$\begin{aligned}&\sum_{j,i=1}^n p_jq_i[h(A_j)f(A_j)g(A_j)]\otimes k(B_i)+\sum_{j,i=1}^n p_jq_ip_jq_ih(A_j)\otimes[k(B_i)f(B_i)g(B_i)]\\&\geq\sum_{j,i=1}^n p_jq_i[h(A_j)f(A_j)]\otimes[k(B_i)g(B_i)]+\sum_{j,i=1}^n p_jq_i[h(A_j)g(A_j)]\otimes[k(B_i)f(B_i)]\end{aligned}$$

and by using the properties of tensorial product we derive (2.9). □

Remark 2.5. *If we take $B_i=A_i$ and $p_i=q_i$, $i\in\{1,\ldots,n\}$, then we get*

$$\begin{aligned}\left(\sum_{i=1}^n p_if(A_i)g(A_i)\right)\otimes 1+1\otimes\left(\sum_{i=1}^n p_if(A_i)g(A_i)\right)\geq&\left(\sum_{i=1}^n p_if(A_i)\right)\otimes\left(\sum_{i=1}^n p_ig(A_i)\right)\\&+\left(\sum_{i=1}^n p_ig(A_i)\right)\otimes\left(\sum_{i=1}^n p_if(A_i)\right),\end{aligned}\tag{2.12}$$

where f, g are synchronous and continuous on I and A_i are selfadjoint with spectra $Sp(A_i)\subset I$, $p_i\geq 0$ for $i\in\{1,\ldots,n\}$ and $\sum_{i=1}^n p_i=1$. By (2.12) we also have the inequality for the Hadamard product

$$\left(\sum_{i=1}^n p_if(A_i)g(A_i)\right)\circ 1\geq\left(\sum_{i=1}^n p_if(A_i)\right)\circ\left(\sum_{i=1}^n p_ig(A_i)\right),\tag{2.13}$$

where f, g are synchronous and continuous on I and A_i are selfadjoint with spectra $Sp(A_i) \subset I$, $p_i \geq 0$ for $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^{n} p_i = 1$.

We also have:

Theorem 2.6. *Let $f, g : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be continuous on $[m,M]$ and differentiable on (m,M) with $g'(t) \neq 0$ for $t \in (m,M)$. Assume that*

$$-\infty < \gamma = \inf_{t\in(m,M)} \frac{f'(t)}{g'(t)}, \quad \sup_{t\in(m,M)} \frac{f'(t)}{g'(t)} = \Gamma < \infty,$$

and A, B selfadjoint operators with spectra $Sp(A), Sp(B) \subseteq [m,M]$, then for any continuous and nonnegative function h defined on $[m,M]$,

$$\begin{aligned}
&\gamma\left[\left(h(A)g^2(A)\right)\otimes h(B) + h(A)\otimes\left(h(B)g^2(B)\right) \ -2\left(g(A)h(A)\right)\otimes\left(h(B)g(B)\right)\right] \\
&\leq [h(A)f(A)g(A)]\otimes h(B) + h(A)\otimes[h(B)f(B)g(B)] - [h(A)f(A)]\otimes[h(B)g(B)] - [h(A)g(A)]\otimes[h(B)f(B)] \\
&\leq \Gamma\left[\left(h(A)g^2(A)\right)\otimes h(B) + h(A)\otimes\left(h(B)g^2(B)\right) \ -2\left(g(A)h(A)\right)\otimes\left(h(B)g(B)\right)\right].
\end{aligned} \tag{2.14}$$

In particular,

$$\begin{aligned}
\gamma\left[g^2(A)\otimes 1 + 1\otimes g^2(B) - 2g(A)\otimes g(B)\right] &\leq [f(A)g(A)]\otimes 1 + 1\otimes[f(B)g(B)] - f(A)\otimes g(B) - g(A)\otimes f(B) \\
&\leq \Gamma\left[g^2(A)\otimes 1 + 1\otimes g^2(B) - 2g(A)\otimes g(B)\right].
\end{aligned} \tag{2.15}$$

Proof. Using the Cauchy mean value theorem, for all $t, s \in [m,M]$ with $t \neq s$ there exists ξ between t and s such that

$$\frac{f(t)-f(s)}{g(t)-g(s)} = \frac{f'(\xi)}{g'(\xi)} \in [\gamma, \Gamma].$$

Therefore

$$\gamma[g(t)-g(s)]^2 \leq [f(t)-f(s)][g(t)-g(s)] \leq \Gamma[g(t)-g(s)]^2$$

for all $t, s \in [m,M]$, which is equivalent to

$$\gamma\left[g^2(t) - 2g(t)g(s) + g^2(s)\right] \leq f(t)g(t) + f(s)g(s) - f(t)g(s) - f(s)g(t) \leq \Gamma\left[g^2(t) - 2g(t)g(s) + g^2(s)\right]$$

for all $t, s \in [m,M]$. If we multiply by $h(t)h(s) \geq 0$, then we get

$$\begin{aligned}
\gamma\left[h(t)g^2(t)h(s) - 2g(t)h(t)h(s)g(s) + h(t)h(s)g^2(s)\right] &\leq h(t)f(t)g(t)h(s) + h(t)h(s)f(s)g(s) \\
&\quad - h(t)f(t)h(s)g(s) - h(t)g(t)h(s)f(s) \\
&\leq \Gamma\left[h(t)g^2(t)h(s) - 2g(t)h(t)h(s)g(s) + h(t)h(s)g^2(s)\right]
\end{aligned}$$

for all $t, s \in [m,M]$.

This implies that

$$\begin{aligned}
&\gamma\int_m^M\int_m^M \left[h(t)g^2(t)h(s) - 2g(t)h(t)h(s)g(s) + h(t)h(s)g^2(s)\right] \times dE(t)\otimes dF(s) \\
&\leq \int_m^M\int_m^M [h(t)f(t)g(t)h(s) + h(t)h(s)f(s)g(s) \ - h(t)f(t)h(s)g(s) - h(t)g(t)h(s)f(s)]\, dE(t)\otimes dF(s) \\
&\leq \Gamma\int_m^M\int_m^M \left[h(t)g^2(t)h(s) - 2g(t)h(t)h(s)g(s) + h(t)h(s)g^2(s)\right] \times dE(t)\otimes dF(s)
\end{aligned}$$

and by performing the calculations as in the proof of Theorem 2.1, we derive (2.14). □

Corollary 2.7. *With the assumptions of Theorem 2.6 we have*

$$\begin{aligned}
&\gamma\left[h(B)\circ\left(h(A)g^2(A)\right) + h(A)\circ\left(h(B)g^2(B)\right) \ -2\left(g(A)h(A)\right)\circ\left(h(B)g(B)\right)\right] \\
&\leq h(B)\circ[h(A)f(A)g(A)] + h(A)\circ[h(B)f(B)g(B)] - [h(A)f(A)]\circ[h(B)g(B)] - [h(A)g(A)]\circ[h(B)f(B)] \\
&\leq \Gamma\left[h(B)\circ\left(h(A)g^2(A)\right) + h(A)\circ\left(h(B)g^2(B)\right) \ -2\left(g(A)h(A)\right)\circ\left(h(B)g(B)\right)\right].
\end{aligned} \tag{2.16}$$

In particular,

$$\gamma\left[\left[g^{2}(A)+g^{2}(B)\right]\circ 1-2g(A)\circ g(B)\right]\leq\left[f(A)g(A)+\left[f(B)g(B)\right]\right]\circ 1-f(A)\circ g(B)-g(A)\circ f(B)$$
$$\leq\Gamma\left[\left[g^{2}(A)+g^{2}(B)\right]\circ 1-2g(A)\circ g(B)\right]. \tag{2.17}$$

We also have:

Corollary 2.8. *With the assumptions of Theorem 2.6 and if A_j are selfadjoint with spectra $Sp(A_j)\subset I$ and $p_j\geq 0,\ j\in\{1,\ldots,n\}$, with $\sum_{j=1}^{n}p_j=1$, then*

$$\begin{aligned}
&\gamma\left\{\left(\sum_{i=1}^{n}p_i g^{2}(A_i)\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_i g^{2}(A_i)\right)-2\left(\sum_{i=1}^{n}p_i g(A_i)\right)\otimes\left(\sum_{i=1}^{n}p_i g(A_i)\right)\right\}\\
&\leq\left(\sum_{i=1}^{n}p_i f(A_i)g(A_i)\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_i f(A_i)g(A_i)\right)-\left(\sum_{i=1}^{n}p_i f(A_i)\right)\otimes\left(\sum_{i=1}^{n}p_i g(A_i)\right)\\
&-\left(\sum_{i=1}^{n}p_i g(A_i)\right)\otimes\left(\sum_{i=1}^{n}p_i f(A_i)\right)\\
&\leq\Gamma\left\{\left(\sum_{i=1}^{n}p_i g^{2}(A_i)\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_i g^{2}(A_i)\right)-2\left(\sum_{i=1}^{n}p_i g(A_i)\right)\otimes\left(\sum_{i=1}^{n}p_i g(A_i)\right)\right\}.
\end{aligned}\tag{2.18}$$

Also,

$$\begin{aligned}
&\gamma\left[\left(\sum_{i=1}^{n}p_i g^{2}(A_i)\right)\circ 1-\left(\sum_{i=1}^{n}p_i g(A_i)\right)\circ\left(\sum_{i=1}^{n}p_i g(A_i)\right)\right]\\
&\leq\left(\sum_{i=1}^{n}p_i f(A_i)g(A_i)\right)\circ 1-\left(\sum_{i=1}^{n}p_i f(A_i)\right)\circ\left(\sum_{i=1}^{n}p_i g(A_i)\right)\\
&\leq\Gamma\left[\left(\sum_{i=1}^{n}p_i g^{2}(A_i)\right)\circ 1-\left(\sum_{i=1}^{n}p_i g(A_i)\right)\circ\left(\sum_{i=1}^{n}p_i g(A_i)\right)\right].
\end{aligned}\tag{2.19}$$

Proof. From (2.15) we get

$$\begin{aligned}
\gamma\left[g^{2}(A_i)\otimes 1+1\otimes g^{2}(A_j)-2g(A_i)\otimes g(A_j)\right]&\leq\left[f(A_i)g(A_i)\right]\otimes 1+1\otimes\left[f(A_j)g(A_j)\right]\\
&-f(A_i)\otimes g(A_j)-g(A_i)\otimes f(A_j)\\
&\leq\Gamma\left[g^{2}(A_i)\otimes 1+1\otimes g^{2}(A_j)-2g(A_i)\otimes g(A_j)\right]
\end{aligned}$$

for all $i,j\in\{1,\ldots,n\}$. If we multiply by $p_ip_j\geq 0$ and sum, then we get

$$\begin{aligned}
\gamma\sum_{i,j=1}^{n}p_ip_j\left[g^{2}(A_i)\otimes 1+1\otimes g^{2}(A_j)-2g(A_i)\otimes g(A_j)\right]&\leq\sum_{i,j=1}^{n}p_ip_j\left\{\left[f(A_i)g(A_i)\right]\otimes 1+1\otimes\left[f(A_j)g(A_j)\right]\right.\\
&\left.-f(A_i)\otimes g(A_j)-g(A_i)\otimes f(A_j)\right\}\\
&\leq\Gamma\sum_{i,j=1}^{n}p_ip_j\left[g^{2}(A_i)\otimes 1+1\otimes g^{2}(A_j)-2g(A_i)\otimes g(A_j)\right],
\end{aligned}$$

which gives (2.18). □

3. Some Examples

Let either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$ and $r\in\mathbb{R}$. If $A,\ B>0$, then from (2.4) we get

$$A^{r+p+q}\otimes B^{r}+A^{r}\otimes B^{r+p+q}\geq A^{r+p}\otimes B^{r+q}+A^{r+q}\otimes B^{r+p}, \tag{3.1}$$

while from (2.6) we obtain

$$A^{r+p+q}\circ B^{r}+A^{r}\circ B^{r+p+q}\geq A^{r+p}\circ B^{r+q}+A^{r+q}\circ B^{r+p}. \tag{3.2}$$

If one of the parameters p,q is in $(-\infty,0)$ while the other in $(0,\infty)$, then the inequality reverses in (3.1) and (3.2).
If we take $q=p$, then we get

$$A^{r+2p}\otimes B^{r}+A^{r}\otimes B^{r+2p}\geq 2A^{r+p}\otimes B^{r+p}, \tag{3.3}$$

and

$$A^{r+2p}\circ B^{r}+A^{r}\circ B^{r+2p}\geq 2A^{r+p}\circ B^{r+p} \tag{3.4}$$

for $p,r\in\mathbb{R}$ and $A,\,B>0$.
If we take $q=-p$, then we get

$$2A^{r}\otimes B^{r}\geq A^{r+p}\otimes B^{r-p}+A^{r-p}\otimes B^{r+p}, \tag{3.5}$$

while from (2.6) we obtain

$$2A^{r}\circ B^{r}\geq A^{r+p}\circ B^{r-p}+A^{r-p}\circ B^{r+p}, \tag{3.6}$$

for $p,r\in\mathbb{R}$ and $A,\,B>0$.
Assume that $A_j>0,\ p_j\geq 0,\ j\in\{1,\ldots,n\}$ with $\sum_{j=1}^{n}p_j=1$, then by (2.12) we get

$$\left(\sum_{i=1}^{n}p_iA_i^{p+q}\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_iA_i^{p+q}\right)\geq\left(\sum_{i=1}^{n}p_iA_i^{p}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{q}\right)+\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{p}\right), \tag{3.7}$$

if either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$. If one of the parameters p,q is in $(-\infty,0)$ while the other in $(0,\infty)$, then the inequality reverses in (3.7). In particular, we derive

$$\left(\sum_{i=1}^{n}p_iA_i^{2p}\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_iA_i^{2p}\right)\geq\left(\sum_{i=1}^{n}p_iA_i^{p}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{p}\right) \tag{3.8}$$

and

$$2\geq\left(\sum_{i=1}^{n}p_iA_i^{p}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{-p}\right)+\left(\sum_{i=1}^{n}p_iA_i^{-p}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{p}\right). \tag{3.9}$$

From (2.13) we obtain

$$\left(\sum_{i=1}^{n}p_iA_i^{p+q}\right)\circ 1\geq\left(\sum_{i=1}^{n}p_iA_i^{p}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{q}\right), \tag{3.10}$$

if either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$. If one of the parameters p,q is in $(-\infty,0)$ while the other in $(0,\infty)$, then the inequality reverses in (3.10). In particular, we have

$$\left(\sum_{i=1}^{n}p_iA_i^{2p}\right)\circ 1\geq\left(\sum_{i=1}^{n}p_iA_i^{p}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{p}\right) \tag{3.11}$$

and

$$1\geq\left(\sum_{i=1}^{n}p_iA_i^{p}\right)\circ\left(\sum_{i=1}^{n}p_iA_i^{-p}\right), \tag{3.12}$$

for $p\in\mathbb{R}$, $A_j>0,\ p_j\geq 0,\ j\in\{1,\ldots,n\}$ with $\sum_{j=1}^{n}p_j=1$.
Consider the functions $f(t)=t^p,\ g(t)=t^q$ defined on $(0,\infty)$. Then $f'(t)=pt^{p-1},\ g'(t)=qt^{q-1}$ for $t>0$ and

$$\frac{f'(t)}{g'(t)}=\frac{p}{q}t^{p-q},\ t>0.$$

Assume that either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$. Then $\frac{p}{q}>0$ and $\frac{f'(t)}{g'(t)}$ is increasing for $p>q$ and decreasing for $p<q$ and constant 1 for $p=q$.
Assume that $0<m\leq A,\, B\leq M$, then

$$\inf_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}m^{p-q} \text{ and } \sup_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}M^{p-q} \text{ for } p>q$$

and

$$\inf_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}M^{p-q} \text{ and } \sup_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}m^{p-q} \text{ for } p<q.$$

Assume that either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$ and $0<m\leq A,\, B\leq M$. From (2.15) we get for $p>q$ that

$$\begin{aligned}
0 &\leq \frac{p}{q}m^{p-q}\left(A^{2q}\otimes 1+1\otimes B^{2q}-2A^{q}\otimes B^{q}\right)\\
&\leq A^{p+q}\otimes 1+1\otimes B^{p+q}-A^{p}\otimes B^{q}-A^{q}\otimes B^{p}\\
&\leq \frac{p}{q}M^{p-q}\left(A^{2q}\otimes 1+1\otimes B^{2q}-2A^{q}\otimes B^{q}\right)
\end{aligned}\tag{3.13}$$

and for $p<q$

$$\begin{aligned}
0 &\leq \frac{p}{q}M^{p-q}\left(A^{2q}\otimes 1+1\otimes B^{2q}-2A^{q}\otimes B^{q}\right)\\
&\leq A^{p+q}\otimes 1+1\otimes B^{p+q}-A^{p}\otimes B^{q}-A^{q}\otimes B^{p}\\
&\leq \frac{p}{q}m^{p-q}\left(A^{2q}\otimes 1+1\otimes B^{2q}-2A^{q}\otimes B^{q}\right).
\end{aligned}\tag{3.14}$$

From (2.17) we also have the inequalities for the Hadamard product for $p>q$ that

$$\begin{aligned}
0 &\leq \frac{p}{q}m^{p-q}\left(\left(A^{2q}+B^{2q}\right)\circ 1-2A^{q}\circ B^{q}\right)\\
&\leq \left(A^{p+q}+B^{p+q}\right)\circ 1-A^{p}\circ B^{q}-A^{q}\circ B^{p}\\
&\leq \frac{p}{q}M^{p-q}\left(\left(A^{2q}+B^{2q}\right)\circ 1-2A^{q}\circ B^{q}\right)
\end{aligned}\tag{3.15}$$

and for $p<q$

$$\begin{aligned}
0 &\leq \frac{p}{q}M^{p-q}\left(\left(A^{2q}+B^{2q}\right)\circ 1-2A^{q}\circ B^{q}\right)\\
&\leq \left(A^{p+q}+B^{p+q}\right)\circ 1-A^{p}\circ B^{q}-A^{q}\circ B^{p}\\
&\leq \frac{p}{q}m^{p-q}\left(\left(A^{2q}+B^{2q}\right)\circ 1-2A^{q}\circ B^{q}\right).
\end{aligned}\tag{3.16}$$

Assume that either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$ and $0<m\leq A_j\leq M$, $p_j\geq 0$, $j\in\{1,\ldots,n\}$ with $\sum_{j=1}^{n}p_j=1$. By (2.18) we get for $p>q$

$$\begin{aligned}
0 &\leq \frac{p}{q}m^{p-q}\left\{\left(\sum_{i=1}^{n}p_iA_i^{2q}\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_iA_i^{2q}\right)-2\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\right\}\\
&\leq \left(\sum_{i=1}^{n}p_iA_i^{p+q}\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_iA_i^{p+q}\right)-\left(\sum_{i=1}^{n}p_iA_i^{p}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{q}\right)-\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{p}\right)\\
&\leq \frac{p}{q}M^{p-q}\left\{\left(\sum_{i=1}^{n}p_iA_i^{2q}\right)\otimes 1+1\otimes\left(\sum_{i=1}^{n}p_iA_i^{2q}\right)-2\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\otimes\left(\sum_{i=1}^{n}p_iA_i^{q}\right)\right\}
\end{aligned}\tag{3.17}$$

and for $p < q$

$$\begin{aligned} 0 &\le \frac{p}{q}M^{p-q}\left\{\left(\sum_{i=1}^{n} p_i A_i^{2q}\right)\otimes 1 + 1\otimes\left(\sum_{i=1}^{n} p_i A_i^{2q}\right) - 2\left(\sum_{i=1}^{n} p_i A_i^{q}\right)\otimes\left(\sum_{i=1}^{n} p_i A_i^{q}\right)\right\} \\ &\le \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right)\otimes 1 + 1\otimes\left(\sum_{i=1}^{n} p_i A_i^{p+q}\right) - \left(\sum_{i=1}^{n} p_i A_i^{p}\right)\otimes\left(\sum_{i=1}^{n} p_i A_i^{q}\right) - \left(\sum_{i=1}^{n} p_i A_i^{q}\right)\otimes\left(\sum_{i=1}^{n} p_i A_i^{p}\right) \\ &\le \frac{p}{q}m^{p-q}\left\{\left(\sum_{i=1}^{n} p_i A_i^{2q}\right)\otimes 1 + 1\otimes\left(\sum_{i=1}^{n} p_i A_i^{2q}\right) - 2\left(\sum_{i=1}^{n} p_i A_i^{q}\right)\otimes\left(\sum_{i=1}^{n} p_i A_i^{q}\right)\right\}. \end{aligned} \tag{3.18}$$

Also, by (2.19) we get for $p > q$

$$\begin{aligned} 0 &\le \frac{p}{q}m^{p-q}\left[\left(\sum_{i=1}^{n} p_i A_i^{2q}\right)\circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q}\right)\circ\left(\sum_{i=1}^{n} p_i A_i^{q}\right)\right] \\ &\le \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right)\circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{p}\right)\circ\left(\sum_{i=1}^{n} p_i A_i^{q}\right) \\ &\le \frac{p}{q}M^{p-q}\left[\left(\sum_{i=1}^{n} p_i A_i^{2q}\right)\circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q}\right)\circ\left(\sum_{i=1}^{n} p_i A_i^{q}\right)\right], \end{aligned} \tag{3.19}$$

while for $p < q$

$$\begin{aligned} 0 &\le \frac{p}{q}M^{p-q}\left[\left(\sum_{i=1}^{n} p_i A_i^{2q}\right)\circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q}\right)\circ\left(\sum_{i=1}^{n} p_i A_i^{q}\right)\right] \\ &\le \left(\sum_{i=1}^{n} p_i A_i^{p+q}\right)\circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{p}\right)\circ\left(\sum_{i=1}^{n} p_i A_i^{q}\right) \\ &\le \frac{p}{q}m^{p-q}\left[\left(\sum_{i=1}^{n} p_i A_i^{2q}\right)\circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{q}\right)\circ\left(\sum_{i=1}^{n} p_i A_i^{q}\right)\right]. \end{aligned} \tag{3.20}$$

Consider the exponential functions $f(t) = \exp(\alpha t)$, $g(t) = \exp(\beta t)$ with $\alpha, \beta \in \mathbb{R}$. If $\alpha\beta > 0$ then the functions have the same monotonicity. If $\alpha\beta < 0$ they have different monotonicity.
If $\alpha\beta > 0$ and A, B are selfadjoint operators, then by (2.5) we get

$$\exp[(\alpha+\beta)A]\otimes 1 + 1\otimes\exp[(\alpha+\beta)B] \ge \exp(\alpha A)\otimes\exp(\beta B) + \exp(\beta A)\otimes\exp(\alpha B), \tag{3.21}$$

and

$$\exp[(\alpha+\beta)A]\circ 1 + 1\circ\exp[(\alpha+\beta)B] \ge \exp(\alpha A)\circ\exp(\beta B) + \exp(\beta A)\circ\exp(\alpha B). \tag{3.22}$$

If $\alpha\beta < 0$, then the reverse inequality holds in (3.21) and (3.22).
If we take $f(t) = t^p$ and $g(t) = \ln t$, we also have the logarithmic inequalities

$$(A^p \ln A)\otimes 1 + 1\otimes(B^p \ln B) \ge A^p \otimes \ln B + \ln A \otimes B^p, \tag{3.23}$$

and

$$(A^p \ln A + B^p \ln B)\circ 1 \ge A^p \circ \ln B + \ln A \circ B^p, \tag{3.24}$$

for A, $B > 0$ and $p > 0$. If $p < 0$, then the inequality reverses in (3.23) and (3.24).

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