# *Some additive reverses of Callebaut and Hölder inequalities for isotonic functionals*

CONSTRUCTIVE MATHEMATICAL ANALYSIS
**6** (2023), No. 4, pp. 249-259
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

*Research Article*

# Some additive reverses of Callebaut and Hölder inequalities for isotonic functionals

SEVER S. DRAGOMIR*

ABSTRACT. In this paper, we obtain some reverses of Callebaut and Hölder inequalities for isotonic functionals via a reverse of Young's inequality we have established recently. Applications for integrals and $n$-tuples of real numbers are provided as well.

**Keywords:** Isotonic functionals, Hölder's inequality, Schwarz's inequality, Callebaut's inequality, integral inequalities, discrete inequalities.

**2020 Mathematics Subject Classification:** 26D15, 26D10.

## 1. INTRODUCTION

Let $L$ be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties:

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$.
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1, t \in E$ then $f_0 \in L$.

An isotonic linear functional $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
(A3) The mapping $A$ is said to be normalised if $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [20] and [21]). For other inequalities for isotonic functionals, see [1], [4]-[19] and [22]-[25]. For related results, see [10, 11]

We note that common examples of such isotonic linear functionals $A$ are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where $\mu$ is a positive measure on $E$ in the first case and $E$ is a subset of the natural numbers $\mathbb{N}$ in the second ($p_k \geq 0$, $k \in E$). As is known to all, the famous Young inequality for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called $\nu$-weighted arithmetic-geometric mean inequality. We consider the function $f_\nu : [0, \infty) \to [0, \infty)$ defined for $\nu \in (0, 1)$

Received: 19.09.2023; Accepted: 28.11.2023; Published Online: 30.11.2023
*Corresponding author: Sever S. Dragomir; sever.dragomir@vu.edu.au
DOI: 10.33205/cma.1362691

by

$$f_{\nu}(x)=1-\nu+\nu x-x^{\nu}. \tag{1.2}$$

For $[m,M]\subset[0,\infty)$, define

$$\Delta_{\nu}(m,M):=\begin{cases} f_{\nu}(m), & M<1\\ \max\{f_{\nu}(m),f_{\nu}(M)\}, & m\le 1\le M\\ f_{\nu}(M), & 1<m\end{cases} \tag{1.3}$$

and

$$\delta_{\nu}(m,M):=\begin{cases} f_{\nu}(M), & M<1\\ 0, & m\le 1\le M\\ f_{\nu}(m), & 1<m\end{cases}. \tag{1.4}$$

In the recent paper [9], we obtained the following refinement and reverse for the additive Young's inequality:

$$\delta_{\nu}(m,M)\,a\le(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\le\Delta_{\nu}(m,M)\,a \tag{1.5}$$

for positive numbers $a$, $b$ with $\frac{b}{a}\in[m,M]\subset(0,\infty)$ and $\nu\in[0,1]$, where $\Delta_{\nu}(m,M)$ and $\delta_{\nu}(m,M)$ are defined by (1.3) and (1.4), respectively.

Kittaneh and Manasrah [16], [17] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2}\le(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\le R\left(\sqrt{a}-\sqrt{b}\right)^{2}, \tag{1.6}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.6) to an identity. Using (1.5) and (1.6), we have the simpler, however coarser bounds:

$$\begin{aligned} & r\times\begin{cases}\left(1-\sqrt{M}\right)^{2}a, & M<1\\ 0, & m\le1\le M\\ \left(\sqrt{m}-1\right)^{2}a, & 1<m\end{cases}\\ & \le(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\\ & \le R\times\begin{cases}\left(1-\sqrt{m}\right)^{2}a, & M<1\\ \max\left\{\left(1-\sqrt{m}\right)^{2},\left(\sqrt{M}-1\right)^{2}\right\}a, & m\le1\le M\\ \left(\sqrt{M}-1\right)^{2}a, & 1<m\end{cases}. \end{aligned} \tag{1.7}$$

We recall that Specht's ratio is defined by [24]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)}, & h\in(0,1)\cup(1,\infty)\\ 1, & h=1\end{cases}. \tag{1.8}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$. The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\le(1-\nu)\,a+\nu b\le S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.9}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$. The second inequality in (1.3) is due to Tominaga [26], while the first one is due to Furuichi [15]. On making use of (1.5) and (1.9), we have the following lower and upper bounds in terms of Specht's ratio:

$$\begin{aligned} &\begin{cases} [S(M^r)-1]\, M^\nu a, & M<1 \\ 0, & m \le 1 \le M \\ [S(m^r)-1]\, m^\nu a, & 1<m \end{cases} \\ &\le (1-\nu)\, a + \nu b - a^{1-\nu} b^\nu \\ &\le \begin{cases} [S(m)-1]\, m^\nu a, & M<1 \\ \max\{[S(m)-1]\, m^\nu, [S(M)-1]\, M^\nu\}\, a, & m \le 1 \le M \\ [S(M)-1]\, M^\nu a, & 1<m \end{cases}. \end{aligned} \tag{1.10}$$

We consider the Kantorovich's constant defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.11}$$

The function $K$ is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h>0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h>0$. The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds.

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^\nu \le (1-\nu)\, a + \nu b \le K^R\left(\frac{a}{b}\right) a^{1-\nu} b^\nu, \tag{1.12}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The first inequality in (1.12) was obtained by Zou et al. in [27], while the second by Liao et al. [18]. By making use of (1.5) and (1.9), we have the following lower and upper bounds in terms of Kantorovich's constant:

$$\begin{aligned} &\begin{cases} [K^r(M)-1]\, M^\nu a, & M<1 \\ 0, & m \le 1 \le M \\ [K^r(m)-1]\, m^\nu a, & 1<m \end{cases} \\ &\le (1-\nu)\, a + \nu b - a^{1-\nu} b^\nu \\ &\le \begin{cases} \left[K^R(m)-1\right] m^\nu a, & M<1 \\ \max\left\{\left[K^R(m)-1\right] m^\nu, \left[K^R(M)-1\right] M^\nu\right\} a, & m \le 1 \le M \\ \left[K^R(M)-1\right] M^\nu a, & 1<m \end{cases}. \end{aligned} \tag{1.13}$$

In this paper, we obtain some reverses of Callebaut and Hölder inequalities for isotonic functionals via the reverse of Young's inequality obtained in (1.5). Applications for integrals and $n$-tuples of real numbers are provided as well.

## 2. Reverses of Callebaut's Inequality

The functional version of Callebaut's inequality states that

$$A^2(fg) \le A\left(f^{2(1-\nu)} g^{2\nu}\right) A\left(f^{2\nu} g^{2(1-\nu)}\right) \le A\left(f^2\right) A\left(g^2\right) \tag{2.14}$$

provided that $f^2$, $g^2$, $f^{2(1-\nu)} g^{2\nu}$, $f^{2\nu} g^{2(1-\nu)}$, $fg \in L$ for some $\nu \in [0,1]$. For the discrete and integral versions in one real variable, see [3].

We start with the following result:

**Theorem 2.1.** *Let $A, B : L \to \mathbb{R}$ be two normalised isotonic functionals. If $f, g : E \to \mathbb{R}$ are such that $f \geq 0$, $g > 0$, $f^2$, $g^2$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and*

$$0 < m \leq \frac{f}{g} \leq M < \infty \tag{2.15}$$

*for some constants $m$, $M$, then*

$$\begin{aligned}(0 \leq)\,& (1-\nu) A\left(f^2\right) B\left(g^2\right) + \nu A\left(g^2\right) B\left(f^2\right) - A\left(f^{2(1-\nu)}g^{2\nu}\right) B\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} A\left(f^2\right) B\left(g^2\right),\end{aligned} \tag{2.16}$$

*where $f_\nu$ is defined by (1.2). In particular,*

$$\begin{aligned}(0 \leq)&A\left(f^2\right) A\left(g^2\right) - A\left(f^{2(1-\nu)}g^{2\nu}\right) A\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} A\left(f^2\right) A\left(g^2\right).\end{aligned} \tag{2.17}$$

*Proof.* For any $x, y \in E$, we have

$$m^2 \leq \frac{f^2(x)}{g^2(x)}, \frac{f^2(y)}{g^2(y)} \leq M^2.$$

Consider

$$a = \frac{f^2(x)}{g^2(x)},\ b = \frac{f^2(y)}{g^2(y)},$$

then $\frac{b}{a} \in \left[\left(\frac{m}{M}\right)^2, \left(\frac{M}{m}\right)^2\right]$ and by the inequality (1.5), we have

$$\begin{aligned}(0 \leq)\,&(1-\nu)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} - \left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu} \\ &\leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\}\frac{f^2(x)}{g^2(x)}\end{aligned} \tag{2.18}$$

for any $x, y \in E$. Now, if we multiply (2.18) by $g^2(x)\,g^2(y) > 0$ then we get

$$\begin{aligned}&(1-\nu) g^2(y) f^2(x) + \nu f^2(y) g^2(x) - f^{2(1-\nu)}(x) g^{2\nu}(x) f^{2\nu}(y) g^{2(1-\nu)}(y) \\ &\leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} f^2(x) g^2(y)\end{aligned} \tag{2.19}$$

for any $x, y \in E$. Fix $y \in E$. Then by (2.19), we have in the order of $L$ that

$$\begin{aligned}&(1-\nu) g^2(y) f^2 + \nu f^2(y) g^2 - f^{2\nu}(y) g^{2(1-\nu)}(y) f^{2(1-\nu)}g^{2\nu} \\ &\leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} g^2(y) f^2.\end{aligned} \tag{2.20}$$

If we take the functional $A$ in (2.19), then we get

$$\begin{aligned}&(1-\nu) g^2(y) A\left(f^2\right) + \nu f^2(y) A\left(g^2\right) - f^{2\nu}(y) g^{2(1-\nu)}(y) A\left(f^{2(1-\nu)}g^{2\nu}\right) \\ &\leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} g^2(y) A\left(f^2\right)\end{aligned}$$

for any $y \in E$. This inequality can be written in the order of $L$ as

$$\begin{aligned}(2.21)\qquad &(1-\nu) A\left(f^2\right) g^2+\nu A\left(g^2\right) f^2-A\left(f^{2(1-\nu)} g^{2\nu}\right) f^{2\nu} g^{2(1-\nu)} \\ &\leq \max \left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} A\left(f^2\right) g^2.\end{aligned}$$

Now, if we take the functional $B$ in (2.21), then we get the desired result (2.16). □

**Corollary 2.1.** *Let $A, B: L \rightarrow \mathbb{R}$ be two normalised isotonic functionals. If $f, g: E \rightarrow \mathbb{R}$ are such that $f \geq 0$, $g>0$, $f^2, g^2, fg \in L$ and the condition (2.15) holds true, then*

$$\begin{aligned}(2.22)\qquad &(0 \leq) \frac{1}{2}\left[A\left(f^2\right) B\left(g^2\right)+A\left(g^2\right) B\left(f^2\right)\right]-A(fg) B(fg) \\ &\leq \frac{1}{2}\left(\frac{M}{m}-1\right)^2 A\left(f^2\right) B\left(g^2\right).\end{aligned}$$

*In particular,*

$$(2.23)\qquad (0 \leq) A\left(f^2\right) A\left(g^2\right)-A^2(fg) \leq \frac{1}{2}\left(\frac{M}{m}-1\right)^2 A\left(f^2\right) A\left(g^2\right),$$

*or, equivalently*

$$(2.24)\qquad (0 \leq) 1-\frac{A^2(fg)}{A\left(f^2\right) A\left(g^2\right)} \leq \frac{1}{2}\left(\frac{M}{m}-1\right)^2.$$

*Proof.* Observe that

$$f_{\frac{1}{2}}\left(\left(\frac{m}{M}\right)^2\right)=\frac{m^2+M^2}{2M^2}-\frac{m}{M}=\frac{(M-m)^2}{2M^2}$$

and

$$f_\nu\left(\left(\frac{M}{m}\right)^2\right)=\frac{m^2+M^2}{2m^2}-\frac{M}{m}=\frac{(M-m)^2}{2m^2}.$$

Therefore

$$\max \left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\}=\frac{(M-m)^2}{2m^2}=\frac{1}{2}\left(\frac{M}{m}-1\right)^2$$

and by (2.16), we get the desired result (2.22). □

**Remark 2.1.** *We observe that the inequality (2.23) can be written as*

$$(2.25)\qquad A\left(f^2\right) A\left(g^2\right)\left[1-\frac{1}{2}\left(\frac{M}{m}-1\right)^2\right] \leq A^2(fg).$$

*We observe that the function $\varphi:[1, \infty) \rightarrow \mathbb{R}$, $\varphi(t)=1-\frac{1}{2}(t-1)^2$ is positive for $t \in (1,1+\sqrt{2})$ and negative for $t \in [1, \infty)$. Therefore, the inequality (2.25) is of interest only in the case that $\frac{M}{m} \in (1,1+\sqrt{2})$.*

On using the inequality (2.16) and (1.7), we get

$$\begin{aligned}(2.26)\qquad &(0 \leq)(1-\nu) A\left(f^2\right) B\left(g^2\right)+\nu A\left(g^2\right) B\left(f^2\right)-A\left(f^{2(1-\nu)} g^{2\nu}\right) B\left(f^{2\nu} g^{2(1-\nu)}\right) \\ &\leq R \max \left\{\left(1-\frac{m}{M}\right)^2,\left(\frac{M}{m}-1\right)^2\right\} A\left(f^2\right) B\left(g^2\right)\end{aligned}$$

and since

$$\max\left\{\left(1-\frac{m}{M}\right)^2,\left(\frac{M}{m}-1\right)^2\right\}=\left(\frac{M}{m}-1\right)^2,$$

then we get from (2.26) that

$$(0\le)\,(1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right)\le R\left(\frac{M}{m}-1\right)^2 A\left(f^2\right)B\left(g^2\right) \tag{2.27}$$

provided $f\ge 0$, $g>0$, $f^2$, $g^2$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$.

On using the inequality (2.16) and (1.10), we get the following reverse of Callebaut's inequality in terms of Specht's ratio

$$(0\le)\,(1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right)\le\max\left\{\left[S\left(\left(\frac{m}{M}\right)^2\right)-1\right]\left(\frac{m}{M}\right)^{2\nu},\left[S\left(\left(\frac{M}{m}\right)^2\right)-1\right]\left(\frac{M}{m}\right)^{2\nu}\right\}A\left(f^2\right)B\left(g^2\right) \tag{2.28}$$

provided $f\ge 0$, $g>0$, $f^2$, $g^2$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$.

Finally, on using the inequality (2.16) and (1.13), we get the following reverse of Callebaut's inequality in terms of Kantorovich's constant

$$(0\le)\,(1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right)\le\max\left\{\left[K^R\left(\left(\frac{m}{M}\right)^2\right)-1\right]\left(\frac{m}{M}\right)^{2\nu},\left[K^R\left(\left(\frac{M}{m}\right)^2\right)-1\right]\left(\frac{M}{m}\right)^{2\nu}\right\}\times A\left(f^2\right)B\left(g^2\right) \tag{2.29}$$

provided $f\ge 0$, $g>0$, $f^2$, $g^2$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$.

## 3. Reverses of Hölder's Inequality

We have the following additive reverse of Hölder's inequality:

**Theorem 3.2.** *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f$, $g:E\to\mathbb{R}$ are such that $fg$, $f^p$, $g^q\in L$ and*

$$0<m_1\le f\le M_1<\infty,\ 0<m_2\le g\le M_2<\infty, \tag{3.30}$$

*then*

$$(0\le)1-\frac{A(fg)}{\left[A\left(f^p\right)\right]^{1/p}\left[A\left(g^q\right)\right]^{1/q}}\le\max\left\{f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right),f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\right\}, \tag{3.31}$$

*where $f_{\frac{1}{p}}$ is defined by*

$$f_{\frac{1}{p}}(x)=\frac{1}{q}+\frac{1}{p}x-x^{\frac{1}{p}}. \tag{3.32}$$

*Proof.* Observe that, by (3.30), we have

$$m_1^p \le A(f^p) \le M_1^p \text{ and } m_2^q \le A(g^q) \le M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p \le \frac{f^p}{A(f^p)} \le \left(\frac{M_1}{m_1}\right)^p \text{ and } \left(\frac{m_2}{M_2}\right)^q \le \frac{g^q}{A(g^q)} \le \left(\frac{M_2}{m_2}\right)^q$$

giving that

$$\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1} \le \frac{\frac{f^p}{A(f^p)}}{\frac{g^q}{A(g^q)}} \le \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q.$$

Using the inequality (1.5) for $b = \frac{f^p}{A(f^p)}$, $a = \frac{g^q}{A(g^q)}$, $\nu = \frac{1}{p}$, $M = \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q$ and $m = \left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}$, we have

$$\begin{aligned} (3.33) \qquad 0 &\le \frac{1}{q}\frac{g^q}{A(g^q)} + \frac{1}{p}\frac{f^p}{A(f^p)} - \frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\le \max\left\{ f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\} \frac{g^q}{A(g^q)}. \end{aligned}$$

If we take the functional $A$ in (3.33), then we get

$$\begin{aligned} 0 &\le \frac{1}{q}\frac{A(g^q)}{A(g^q)} + \frac{1}{p}\frac{A(f^p)}{A(f^p)} - \frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\le \max\left\{ f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\} \frac{A(g^q)}{A(g^q)}, \end{aligned}$$

which is equivalent to the desired result (3.30). □

The following reverse of Cauchy-Bunyakovsky-Schwarz inequality for isotonic functionals holds:

**Corollary 3.2.** *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional, $f, g : E \to \mathbb{R}$ are such that $fg$, $f^2$, $g^2 \in L$ and the condition (3.30) is valid, then*

$$(3.34) \qquad (0 \le)\, 1 - \frac{A(fg)}{[A(f^2)]^{1/2}[A(g^2)]^{1/2}} \le \frac{(M_1M_2 - m_1m_2)^2}{2m_1^2m_2^2}.$$

*Proof.* For $p = 2$, we have $f_{\frac{1}{2}}(x) = \frac{1+x}{2} - \sqrt{x}$, $x \ge 0$. Then

$$f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^2 \left(\frac{M_2}{m_2}\right)^2\right) = \frac{(M_1M_2 - m_1m_2)^2}{2m_1^2m_2^2}$$

and

$$f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{-2} \left(\frac{M_2}{m_2}\right)^{-2}\right) = \frac{(M_1M_2 - m_1m_2)^2}{2M_1^2M_2^2}$$

and since

$$\max\left\{ f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^2 \left(\frac{M_2}{m_2}\right)^2\right), f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{-2} \left(\frac{M_2}{m_2}\right)^{-2}\right)\right\} = \frac{(M_1M_2 - m_1m_2)^2}{2m_1^2m_2^2},$$

then by (3.31) we get the desired result (3.34). □

Using the inequality (3.34) and (1.7), we get

$$(0 \leq)\, 1-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \leq T\max\left\{\left(1-\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}}\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right)^2, \left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^2\right\}, \tag{3.35}$$

where $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$. Since

$$\max\left\{\left(1-\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}}\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right)^2, \left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^2\right\} = \left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^2,$$

then by (3.35) we have the inequality

$$(0 \leq)\, 1-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \leq T\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^2, \tag{3.36}$$

where $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$, $f,g:E\to\mathbb{R}$ are such that $fg, f^p, g^q\in L$ and they satisfy the condition (3.30). Using the inequality (3.34) and (1.10), we get

$$\begin{aligned}(0 \leq) 1&-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\leq \max\left\{\left[S\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right)-1\right]\left(\frac{M_1}{m_1}\right)^{-1}\left(\frac{M_2}{m_2}\right)^{-\frac{q}{p}},\right. \\ &\qquad\left.\left[S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)-1\right]\left(\frac{M_1}{m_1}\right)\left(\frac{M_2}{m_2}\right)^{\frac{q}{p}}\right\}\end{aligned} \tag{3.37}$$

provided $f,g:E\to\mathbb{R}$ are such that $fg, f^p, g^q\in L$ and they satisfy the condition (3.30). Using the inequality (3.34) and (1.13), we get

$$\begin{aligned}(0 \leq) 1&-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\leq \max\left\{\left[K^T\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right)-1\right]\left(\frac{M_1}{m_1}\right)^{-1}\left(\frac{M_2}{m_2}\right)^{-\frac{q}{p}},\right. \\ &\qquad\left.\left[K^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)-1\right]\left(\frac{M_1}{m_1}\right)\left(\frac{M_2}{m_2}\right)^{\frac{q}{p}}\right\},\end{aligned} \tag{3.38}$$

where $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$, $f,g:E\to\mathbb{R}$ are such that $fg, f^p, g^q\in L$ and they satisfy the condition (3.30).

## 4. Applications for Integrals

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set $\Omega$, a $\sigma$ -algebra $\mathcal{A}$ of subsets of $\Omega$ and a countably additive and positive measure $\mu$ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a $\mu$-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for $\mu$ -a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)| w(x) d\mu(x) < \infty\}.$$

For simplicity of notation, we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x) d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega w d\mu = 1$.

Let $f, g$ be $\mu$-measurable functions with the property that there exists the constants $M, m > 0$ such that

$$0 < m \leq \frac{f}{g} \leq M < \infty \ \mu\text{-almost everywhere (a.e.) on } \Omega. \tag{4.39}$$

If $f^2, g^2 \in L_w(\Omega, \mu)$, then by (2.17) we have

$$\begin{aligned} (0 \leq) & \int_\Omega w f^2 d\mu \int_\Omega w g^2 d\mu - \int_\Omega w f^{2(1-s)} g^{2s} d\mu \int_\Omega w f^{2s} g^{2(1-s)} d\mu \\ & \leq \max\left\{ f_s\left(\left(\frac{m}{M}\right)^2\right), f_s\left(\left(\frac{M}{m}\right)^2\right)\right\} \int_\Omega w f^2 d\mu \int_\Omega w g^2 d\mu \end{aligned} \tag{4.40}$$

for any $s \in [0,1]$, where $f_s$ is defined by (1.2), and, in particular,

$$(0 \leq)\, 1 - \frac{\left(\int_\Omega w f g d\mu\right)^2}{\int_\Omega w f^2 d\mu \int_\Omega w g^2 d\mu} \leq \frac{1}{2}\left(\frac{M}{m} - 1\right)^2. \tag{4.41}$$

Let $f$, $g$ be $\mu$-measurable functions with the property that there exists the constants $m_1$, $M_1$, $m_2$, $M_2$ such that

$$0 < m_1 \leq f \leq M_1 < \infty,\ 0 < m_2 \leq g \leq M_2 < \infty \ \mu\text{-a.e. on } \Omega. \tag{4.42}$$

Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (3.31) we have the following reverse of Hölder's inequality

$$\begin{aligned} (0 \leq) 1 & - \frac{\int_\Omega w f g d\mu}{\left(\int_\Omega w f^p d\mu\right)^{1/p} \left(\int_\Omega w g^q d\mu\right)^{1/q}} \\ & \leq \max\left\{ f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\}, \end{aligned} \tag{4.43}$$

where $f_{\frac{1}{p}}$ is defined by (3.32).

In particular, we have the reverse of Cauchy-Bunyakovsky-Schwarz inequality

$$(0 \leq)\, 1 - \frac{\int_\Omega w f g d\mu}{\left(\int_\Omega w f^2 d\mu\right)^{1/2} \left(\int_\Omega w g^2 d\mu\right)^{1/2}} \leq \frac{(M_1 M_2 - m_1 m_2)^2}{2 m_1^2 m_2^2}. \tag{4.44}$$

From (3.36), we have, for $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, that

$$(0 \leq)\, 1 - \frac{\int_\Omega w f g d\mu}{\left(\int_\Omega w f^p d\mu\right)^{1/p} \left(\int_\Omega w g^q d\mu\right)^{1/q}} \leq T\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}} \left(\frac{M_2}{m_2}\right)^{\frac{q}{2}} - 1\right)^2. \tag{4.45}$$

## 5. Applications for Real Numbers

We consider the $n$-tuples of positive numbers $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and the probability distribution $p = (p_1, ..., p_n)$, i.e. $p_i \geq 0$ for any $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$.

If there exist the constants $m, M > 0$ such that

$$0 < m \leq \frac{a_i}{b_i} \leq M < \infty \text{ for any } i \in \{1, ..., n\},$$

then by (4.40), for the counting discrete measure, we have

$$\begin{aligned} (5.46) \qquad & (0 \leq) \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i a_i^{2(1-s)} b_i^{2s} \sum_{i=1}^n p_i a_i^{2s} b_i^{2(1-s)} \\ & \leq \max\left\{ f_s\left(\left(\frac{m}{M}\right)^2\right), f_s\left(\left(\frac{M}{m}\right)^2\right)\right\} \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 \end{aligned}$$

for any $s \in [0, 1]$, where $f_s$ is defined by (1.2). In particular,

$$(5.47) \qquad (0 \leq) \, 1 - \frac{\left(\sum_{i=1}^n p_i a_i b_i\right)^2}{\sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2} \leq \frac{1}{2}\left(\frac{M}{m} - 1\right)^2.$$

If there exists the constants $m_1, M_1, m_2, M_2$ such that

$$(5.48) \qquad 0 < m_1 \leq a_i \leq M_1 < \infty, \; 0 < m_2 \leq b_i \leq M_2 < \infty \text{ for any } i \in \{1, ..., n\}$$

and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (4.43) we have the following reverse of Hölder's inequality

$$\begin{aligned} (5.49) \qquad & (0 \leq) 1 - \frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p} \left(\sum_{i=1}^n p_i b_i^q\right)^{1/q}} \\ & \leq \max\left\{ f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\}, \end{aligned}$$

where $f_{\frac{1}{p}}$ is defined by (3.32). In particular, we have the reverse of Cauchy-Bunyakovsky-Schwarz inequality

$$(5.50) \qquad (0 \leq) \, 1 - \frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^2\right)^{1/2} \left(\sum_{i=1}^n p_i b_i^2\right)^{1/2}} \leq \frac{(M_1 M_2 - m_1 m_2)^2}{2 m_1^2 m_2^2}.$$

From (4.45), we have for $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, that

$$(5.51) \qquad (0 \leq) \, 1 - \frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p} \left(\sum_{i=1}^n p_i b_i^q\right)^{1/q}} \leq T\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}} \left(\frac{M_2}{m_2}\right)^{\frac{q}{2}} - 1\right)^2$$

provided $a$ and $b$ satisfy the condition (5.48).

## References


[1] D. Andrica, C. Badea: *Grüss' inequality for positive linear functionals*, Periodica Math. Hung., **19** (1998), 155–167.

[2] P. R. Beesack, J. E. Pečarić: *On Jessen's inequality for convex functions*, J. Math. Anal. Appl., **110** (1985), 536–552.

[3] D. K. Callebaut: *Generalization of Cauchy-Schwarz inequality*, J. Math. Anal. Appl., **12** (1965), 491–494.

[4] S. S. Dragomir: *A refinement of Hadamard's inequality for isotonic linear functionals*, *Tamkang J. Math.*(Taiwan), **24** (1992), 101–106.

[5] S. S. Dragomir: *On a reverse of Jessen's inequality for isotonic linear functionals*, J. Ineq. Pure & Appl. Math., **2** (3) (2001), Article 36.

[6] S. S. Dragomir: *On the Jessen's inequality for isotonic linear functionals*, Nonlinear Anal. Forum, **7** (2) (2002), 139–151.

[7] S. S. Dragomir: *On the Lupaş-Beesack-Pečarić inequality for isotonic linear functionals*, *Nonlinear Funct. Anal. & Appl.*, **7** (2) (2002), 285–298.
[8] S.S. Dragomir: *Bounds for the normalized Jensen functional*, *Bull. Austral. Math. Soc.*, **74** (3) (2006), 417–478.
[9] S. S. Dragomir: *Additive refinements and reverses of Young's operator inequality with applications*, Preprint RGMIA Res. Rep. Coll., **18** (2015), Art. A 165.
[10] S. S. Dragomir: *Inequalities for Synchronous Functions and Applications*, Constr. Math. Anal., **2** (3) (2019), 109–123.
[11] S. S. Dragomir, *Ostrowski's Type Inequalities for the Complex Integral on Paths*, Constr. Math. Anal., **3** (4) (2020), 125–138.
[12] S. S. Dragomir, N. M. Ionescu: *On some inequalities for convex-dominated functions*, *L'Anal. Num. Théor. L'Approx.*, **19** (1) (1990), 21–27.
[13] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications,* RGMIA Monographs, Victoria University (2000).
[14] S. S. Dragomir, C. E. M. Pearce and J. E. Pečarić: *On Jessen's and related inequalities for isotonic sublinear functionals*, Acta. Sci. Math.(Szeged), **61** (1995), 373–382.
[15] S. Furuichi: *Refined Young inequalities with Specht's ratio*, J. Egyptian Math. Soc., **20** (2012), 46–49.
[16] F. Kittaneh, Y. Manasrah: *Improved Young and Heinz inequalities for matrix*, J. Math. Anal. Appl., **361** (2010), 262–269.
[17] F. Kittaneh, Y. Manasrah: *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra, **59** (2011), 1031–1037.
[18] W. Liao, J. Wu and J. Zhao: *New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant*, Taiwanese J. Math., **19** (2) (2015), 467–479.
[19] A. Lupaş: *A generalisation of Hadamard's inequalities for convex functions*, Univ. Beograd. Elek. Fak., **544/576** (1976), 115–121.
[20] J .E. Pečarić: *On Jessen's inequality for convex functions (III)*, J. Math. Anal. Appl., **156** (1991), 231–239.
[21] J. E. Pečarić, P. R. Beesack: *On Jessen's inequality for convex functions (II)*, J. Math. Anal. Appl., **118** (1986), 125–144.
[22] J. E. Pečarić, S. S. Dragomir: *A generalisation of Hadamard's inequality for isotonic linear functionals*, Radovi Mat.(Sarjevo), **7** (1991), 103–107.
[23] J. E. Pečarić, I. Raşa: *On Jessen's inequality*, Acta. Sci. Math.(Szeged), **56** (1992), 305–309.
[24] W. Specht: *Zer Theorie der elementaren Mittel*, Math. Z., **74** (1960), 91–98.
[25] G. Toader, S. S. Dragomir: *Refinement of Jessen's inequality*, Demonstr. Math., **28** (1995), 329–334.
[26] M. Tominaga: *Specht's ratio in the Young inequality*, Sci. Math. Japon., **55** (2002), 583–588.
[27] G. Zuo, G. Shi and M. Fujii: *Refined Young inequality with Kantorovich constant*, J. Math. Inequal., **5** (2011), 551–556.

Sever S. Dragomir
Victoria University
School of Engineering & Science
PO Box 14428, Melbourne City, MC 8001, Australia

University of the Witwatersrand
School of Computer Science & Applied Mathematics
Private Bag 3, Johannesburg 2050, South Africa
ORCID: 0000-0003-2902-6805
*Email address*: sever.dragomir@vu.edu.au