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Tensorial and Hadamard product inequalities for functions of selfadjoint operators in Hilbert spaces in terms of Kantorovich ratio

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Abstract: Let H be a Hilbert space. In this paper we show among others that, if f, g are continuous on the interval I with

$$0 < \gamma \leq \frac{f(t)}{g(t)} \leq \Gamma \qquad \text{for } t \in I$$

and if A and B are selfadjoint operators with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then

$$\begin{aligned}\left[f^{1-\nu}(A)\, g^{\nu}(A)\right] \otimes \left[f^{\nu}(B)\, g^{1-\nu}(B)\right] &\leq (1-\nu) f(A) \otimes g(B) + \nu g(A) \otimes f(B) \\ &\leq \left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R \left[f^{1-\nu}(A)\, g^{\nu}(A)\right] \otimes \left[f^{\nu}(B)\, g^{1-\nu}(B)\right].\end{aligned}$$

The above inequalities also hold for the Hadamard product “$\circ$” instead of tensorial product “$\otimes$”.

Key words: Tensorial product, Hadamard Product, Selfadjoint operators, Convex functions.
MSC (2020): 47A63, 47A99.

1. Introduction

Let $I_1, \ldots, I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times \cdots \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, \ldots, A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, \ldots, H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, \ldots, k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

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is the spectral resolution of A_i for $i = 1, \dots, k$; by following [2], we define

$$f(A_1, \dots, A_k) := \int_{I_1} \cdots \int_{I_k} f(\lambda_1, \dots, \lambda_k)\, dE_1(\lambda_1) \otimes \cdots \otimes dE_k(\lambda_k) \qquad (1.1)$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes \cdots \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [11] for functions of two variables and have the property that

$$f(A_1, \dots, A_k) = f_1(A_1) \otimes \cdots \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, \dots, t_k) = f_1(t_1) \cdots f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq)\, f(s)\, f(t) \qquad \text{for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [13, p. 173]

$$f(A \otimes B) \geq (\leq)\, f(A) \otimes f(B) \qquad \text{for all } A, B \geq 0. \qquad (1.2)$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \qquad \text{and} \qquad B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \qquad (1.3)$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A, B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \qquad \text{and} \qquad (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [16] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B) &\le \frac{1}{2}\left[(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)\right]\\ &\le \frac{1}{2}\left(A\otimes B+B\otimes A\right) \qquad (1.4)\end{aligned}$$

for $A, B > 0$ and $\alpha \in [0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A\circ B)\, e_j, e_j\rangle = \langle Ae_j, e_j\rangle \langle Be_j, e_j\rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A\circ B = \mathcal{U}^*\,(A\otimes B)\,\mathcal{U} \qquad (1.5)$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative operator concave (sub-multiplicative operator convex)* on $[0,\infty)$, then also [13, p. 173]

$$f\,(A\circ B) \ge (\le)\, f\,(A)\circ f\,(B) \qquad \text{for all } A, B \ge 0. \qquad (1.6)$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2} \le \left(\frac{A+B}{2}\right)\circ 1 \qquad \text{for all } A, B \ge 0$$

and *Fiedler inequality*

$$A\circ A^{-1} \ge 1 \text{ for } A > 0. \qquad (1.7)$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B \le \left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2} \qquad \text{for all } A, B \ge 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B \le \left(A^2\circ B^2\right)^{1/2} \qquad \text{for } A, B \ge 0.$$

It has been shown in [10] that $\left(A^2 \circ 1\right)^{1/2}\left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

The famous *Young inequality* for scalars says that, if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)\,a + \nu b \tag{1.8}$$

with equality if and only if $a = b$. The inequality (1.8) is also called ν-weighted arithmetic-geometric mean inequality.

Kittaneh and Manasrah [8, 9] provided a refinement and an additive reverse for Young inequality (1.8) as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \leq (1-\nu)\,a + \nu b - a^{1-\nu}b^{\nu} \leq R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.9}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.9) to an identity and is of no interest.

We recall that *Specht's ratio* is defined by [14]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0,1)\cup(1,\infty), \\ 1 & \text{if } h = 1. \end{cases} \tag{1.10}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function S is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality (1.8)

$$S\left(\left(\frac{a}{b}\right)^r\right)a^{1-\nu}b^{\nu} \leq (1-\nu)\,a + \nu b \leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.11}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.11) is due to Tominaga [15] while the first one is due to Furuichi [7].

It is an open question for the author if in the right hand side of (1.11) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^R\right)$ where $R = \max\{1-\nu, \nu\}$.

Kittaneh and Manasrah result provides upper and lower bounds for the difference between the weighted arithmetic mean and geometric mean while Tominaga and Furuichi results provides bounds for the quotient of these means. They cannot be compared in general.

We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \qquad \text{h} > 0. \tag{1.12}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h>0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^{r}\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \leq K^{R}\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.13}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.13) was obtained by Zuo et al. in [17] while the second by Liao et al. [12].

In [17] the authors also showed that

$$K^{r}(h) \geq S(h^{r}) \qquad \text{for } h>0 \text{ and } r \in \left[0, \frac{1}{2}\right]$$

implying that the lower bound in (1.13) is better than the lower bound from (1.11).

In [5] the authors showed that neither of the upper bounds in (1.11) and (1.13) is always best.

We can give here a simple direct proof for (1.13) as follows.

Recall the following result obtained by the author in 2006 [4] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n \min_{j\in\{1,2,\dots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^{n} \Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)\right] \\ & \leq \frac{1}{P_n}\sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \\ & \leq n \max_{j\in\{1,2,\dots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^{n} \Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)\right], \end{aligned} \tag{1.14}$$

where $\Phi: C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n=2$, we deduce from (1.14) that

$$2\min\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \tag{1.15}$$
$$\leq \nu\Phi(x)+(1-\nu)\Phi(y)-\Phi[\nu x+(1-\nu)y]$$
$$\leq 2\max\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]$$

for any $x,y\in\mathbb{R}$ and $\nu\in[0,1]$.

Now, if we write the inequality (1.15) for the convex function $\Phi(x) = -\ln x$, and for the positive numbers a and b we get (1.13).

Motivated by the above results, in this paper we show among others that, if f,g are continuous on the interval I with

$$0<\gamma\leq\frac{f(t)}{g(t)}\leq\Gamma \qquad \text{for } t\in I$$

and if A and B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$, then

$$\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)\,g^{1-\nu}(B)\right]$$
$$\leq(1-\nu)\,f(A)\otimes g(B)+\nu g(A)\otimes f(B)$$
$$\leq\left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)\,g^{1-\nu}(B)\right].$$

The above inequalities also hold for the Hadamard product "$\circ$" instead of tensorial product "$\otimes$".

2. Main results

We have:

THEOREM 1. *Let I and J be two intervals and f, g defined and continuous on an interval containing $I\cup J$. Assume that*

$$0<\gamma_1\leq\frac{f(t)}{g(t)}\leq\Gamma_1 \qquad \text{for } t\in I$$

and

$$0<\gamma_2\leq\frac{f(s)}{g(s)}\leq\Gamma_2 \qquad \text{for } s\in J.$$

Define

$$U(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2) := \begin{cases} K\left(\frac{\Gamma_1}{\gamma_2}\right) & \text{if } 1 \le \frac{\gamma_1}{\Gamma_2}, \\ \max\left\{K\left(\frac{\Gamma_1}{\gamma_2}\right), K\left(\frac{\gamma_1}{\Gamma_2}\right)\right\} & \text{if } \frac{\gamma_1}{\Gamma_2} < 1 < \frac{\Gamma_1}{\gamma_2}, \\ K\left(\frac{\gamma_1}{\Gamma_2}\right) & \text{if } \frac{\Gamma_1}{\gamma_2} \le 1, \end{cases}$$

and

$$u(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2) = \begin{cases} K\left(\frac{\gamma_1}{\Gamma_2}\right) & \text{if } 1 \le \frac{\gamma_1}{\Gamma_2}, \\ 1 & \text{if } \frac{\gamma_1}{\Gamma_2} < 1 < \frac{\Gamma_1}{\gamma_2}, \\ K\left(\frac{\Gamma_1}{\gamma_2}\right) & \text{if } \frac{\Gamma_1}{\gamma_2} \le 1. \end{cases}$$

If A and B are selfadjoint operators with $\operatorname{Sp}(A) \subset I$ *and* $\operatorname{Sp}(B) \subset J$*, then*

$$\begin{aligned} u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)&\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)\,g^{1-\nu}(B)\right] \\ &\le (1-\nu)\,f(A)\otimes g(B) + \nu g(A)\otimes f(B) \\ &\le U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)\,g^{1-\nu}(B)\right] \end{aligned} \tag{2.1}$$

for $\nu \in [0,1]$*, where* $r = \min\{1-\nu,\nu\}$ *and* $R = \max\{1-\nu,\nu\}$*.*

Proof. If $a \in [\gamma_1,\Gamma_1] \subset (0,\infty)$ and $b \in [\gamma_2,\Gamma_2] \subset (0,\infty)$, then

$$\frac{a}{b} \in \left[\frac{\gamma_1}{\Gamma_2}, \frac{\Gamma_1}{\gamma_2}\right] \subset (0,\infty).$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, then we observe that

$$\max_{\tau\in\left[\frac{\gamma_1}{\Gamma_2},\frac{\Gamma_1}{\gamma_2}\right]} K(\tau) = U(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)$$

and

$$\min_{\tau\in\left[\frac{\gamma_1}{\Gamma_2},\frac{\Gamma_1}{\gamma_2}\right]} K(\tau) = u(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2).$$

By (1.13) we then get

$$\begin{aligned} u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\,a^{1-\nu}b^{\nu} &\le K^r\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \le (1-\nu)\,a + \nu b \\ &\le K^R\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \le U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\,a^{1-\nu}b^{\nu}, \end{aligned} \tag{2.2}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Now, if we take

$$a = \frac{f(t)}{g(t)}, \quad t \in I \qquad \text{and} \qquad b = \frac{f(s)}{g(s)}, \quad s \in J$$

in (2.2), then we get

$$\begin{aligned} u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left(\frac{f(t)}{g(t)}\right)^{1-\nu} \left(\frac{f(s)}{g(s)}\right)^{\nu} & \\ \leq (1-\nu) \frac{f(t)}{g(t)} + \nu \frac{f(s)}{g(s)} & \\ \leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left(\frac{f(t)}{g(t)}\right)^{1-\nu} \left(\frac{f(s)}{g(s)}\right)^{\nu}, & \end{aligned} \tag{2.3}$$

for $t \in I$ and $s \in J$.

This is equivalent to

$$\begin{aligned} u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s) & \\ \leq (1-\nu) f(t) g(s) + \nu g(t) f(s) & \\ \leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s), & \end{aligned} \tag{2.4}$$

for $t \in I$ and $s \in J$.

If

$$A = \int_I t dE(t) \qquad \text{and} \qquad B = \int_J s dF(s)$$

are the spectral resolutions of A and B, then by taking the integral $\int_I \int_J$ over $dE(t) \otimes dF(s)$ in (2.4), we derive that

$$\begin{aligned} u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \int_I \int_J f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s) dE(t) \otimes dF(s) & \\ \leq \int_I \int_J [(1-\nu) f(t) g(s) + \nu g(t) f(s)] dE(t) \otimes dF(s) & \\ \leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \int_I \int_J f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s) dE(t) \otimes dF(s). & \end{aligned} \tag{2.5}$$

By utilizing (1.1) we get

$$\begin{aligned} \int_I \int_J f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s) dE(t) \otimes dF(s) & \\ = \left[f^{1-\nu}(A) g^{\nu}(A)\right] \otimes \left[f^{\nu}(B) g^{1-\nu}(B)\right] & \end{aligned}$$

and

$$\begin{aligned}&\int_I\int_J [(1-\nu)f(t)g(s)+\nu g(t)f(s)]\,dE(t)\otimes dF(s)\\ &\qquad=(1-\nu)\int_I\int_J f(t)g(s)dE(t)\otimes dF(s)+\nu\int_I\int_J g(t)f(s)dE(t)\otimes dF(s)\\ &\qquad=(1-\nu)f(A)\otimes g(B)+\nu g(A)\otimes f(B).\end{aligned}$$

Therefore, by (2.5) we obtain the desired result (2.1). ∎

Corollary 1. *With the assumptions of Theorem 1,*

$$\begin{aligned}u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)&\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(B)\,g^{1-\nu}(B)\right] \qquad (2.6)\\ &\le(1-\nu)\,f(A)\circ g(B)+\nu g(A)\circ f(B)\\ &\le U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(B)\,g^{1-\nu}(B)\right]\end{aligned}$$

for $\nu\in[0,1]$.

Proof. We have the representation

$$X\circ Y=\mathcal{U}^*(X\otimes Y)\,\mathcal{U},$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If we take $\mathcal{U}^*$ at the left and $\mathcal{U}$ at the right in (2.1), then we get

$$\begin{aligned}u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)&\mathcal{U}^*\left(\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)\,g^{1-\nu}(B)\right]\right)\mathcal{U}\\ &\le\mathcal{U}^*\left[(1-\nu)\,f(A)\otimes g(B)+\nu g(A)\otimes f(B)\right]\mathcal{U}\\ &\le U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\,\mathcal{U}^*\left(\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\otimes\left[f^{\nu}(B)\,g^{1-\nu}(B)\right]\right)\mathcal{U},\end{aligned}$$

which is equivalent to

$$\begin{aligned}u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)&\mathcal{U}^*\left(\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(B)\,g^{1-\nu}(B)\right]\right)\mathcal{U}\\ &\le(1-\nu)\,\mathcal{U}^*\left[f(A)\circ g(B)\right]\mathcal{U}+\nu\,\mathcal{U}^*\left[g(A)\circ f(B)\right]\mathcal{U}\\ &\le U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\,\mathcal{U}^*\left(\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(B)\,g^{1-\nu}(B)\right]\right)\mathcal{U}\end{aligned}$$

and the inequality (2.6) is obtained. ∎

Corollary 2. *Assume that* f,g *are continuous on* I *and*

$$0<\gamma\le\frac{f(t)}{g(t)}\le\Gamma\qquad\text{for } t\in I.$$

If A and B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, *then*

$$\begin{aligned} &\left[f^{1-\nu}(A)\, g^{\nu}(A)\right] \otimes \left[f^{\nu}(B)\, g^{1-\nu}(B)\right] \\ &\qquad \leq (1-\nu) f(A) \otimes g(B) + \nu g(A) \otimes f(B) \\ &\qquad \leq \left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R \left[f^{1-\nu}(A)\, g^{\nu}(A)\right] \otimes \left[f^{\nu}(B)\, g^{1-\nu}(B)\right]. \end{aligned} \tag{2.7}$$

We also have for $B = A$ that

$$\begin{aligned} &\left[f^{1-\nu}(A)\, g^{\nu}(A)\right] \otimes \left[f^{\nu}(A)\, g^{1-\nu}(A)\right] \\ &\qquad \leq (1-\nu) f(A) \otimes g(A) + \nu g(A) \otimes f(A) \\ &\qquad \leq \left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R \left[f^{1-\nu}(A)\, g^{\nu}(A)\right] \otimes \left[f^{\nu}(A)\, g^{1-\nu}(A)\right]. \end{aligned} \tag{2.8}$$

The proof follows by taking $\gamma_1 = \gamma_2 = \gamma$ and $\Gamma_1 = \Gamma_2 = \Gamma$ in Theorem 1. We also have:

Theorem 2. *With the assumptions of Theorem 1, we have*

$$\begin{aligned} u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) &\leq (1-\nu)\left[f^{\nu}(A)\, g^{-\nu}(A)\right] \otimes \left[f^{-\nu}(B)\, g^{\nu}(B)\right] \\ &\quad + \nu\left[g^{1-\nu}(A)\, f^{-1+\nu}(A)\right] \otimes \left[g^{-1+\nu}(B)\, f^{1-\nu}(B)\right] \\ &\leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2), \end{aligned} \tag{2.9}$$

for all $\nu \in [0, 1]$.

Proof. From (2.4) we also have

$$\begin{aligned} u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) &\leq \frac{(1-\nu) f(t)\, g(s) + \nu g(t)\, f(s)}{f^{1-\nu}(t)\, g^{\nu}(t)\, f^{\nu}(s)\, g^{1-\nu}(s)} \\ &\leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2), \end{aligned}$$

namely

$$\begin{aligned} u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) &\leq (1-\nu) f^{\nu}(t)\, g^{-\nu}(t)\, f^{-\nu}(s)\, g^{\nu}(s) \\ &\quad + \nu g^{1-\nu}(t)\, f^{-1+\nu}(t)\, g^{-1+\nu}(s)\, f^{1-\nu}(s) \\ &\leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2), \end{aligned} \tag{2.10}$$

for $t \in I$ and $s \in J$.

By taking the integral $\int_I \int_J$ over $dE(t) \otimes dF(s)$ in (2.10), we derive the desired inequality (2.9). ∎

Remark 1. The above inequalities (2.7), (2.8) and (2.9) also hold for the Hadamard product "$\circ$" instead of tensorial product "$\otimes$". They can be proved by making use of a similar argument to the one in the proof of Corollary 1.

3. Inequalities for sums

We can state the following result:

PROPOSITION 1. *With the assumptions of Theorem 1 and if A_i and B_i are selfadjoint operators with $\operatorname{Sp}(A_i) \subset I$ and $\operatorname{Sp}(B_i) \subset J$, $p_i \geq 0$, $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^n p_i = 1$, then*

$$\begin{aligned} u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2) & \left[\sum_{i=1}^n p_i f^{1-\nu}(A_i) g^{\nu}(A_i)\right] \otimes \left[\sum_{i=1}^n p_i f^{\nu}(B_i) g^{1-\nu}(B_i)\right] \\ & \leq (1-\nu) \left(\sum_{i=1}^n p_i f(A_i)\right) \otimes \left(\sum_{i=1}^n p_i g(B_i)\right) \\ & \quad + \nu \left(\sum_{i=1}^n p_i g(A_i)\right) \otimes \left(\sum_{i=1}^n p_i f(B_i)\right) \\ & \leq U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2) \left[\sum_{i=1}^n p_i f^{1-\nu}(A_i) g^{\nu}(A_i)\right] \otimes \left[\sum_{i=1}^n p_i f^{\nu}(B_i) g^{1-\nu}(B_i)\right] \end{aligned} \tag{3.1}$$

for $\nu \in [0,1]$, where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Proof. From (2.1) we get

$$\begin{aligned} u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2) & \left[f^{1-\nu}(A_i) g^{\nu}(A_i)\right] \otimes \left[f^{\nu}(B_j) g^{1-\nu}(B_j)\right] \\ & \leq (1-\nu) f(A_i) \otimes g(B_j) + \nu g(A_i) \otimes f(B_j) \\ & \leq U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2) \left[f^{1-\nu}(A_i) g^{\nu}(A_i)\right] \otimes \left[f^{\nu}(B_j) g^{1-\nu}(B_j)\right] \end{aligned} \tag{3.2}$$

for $i, j \in \{1, \ldots, n\}$.

If we multiply (3.2) by $p_ip_j \geq 0$ and sum, then we get

$$u^r(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\sum_{i,j=1}^{n} p_ip_j\left[f^{1-\nu}(A_i)\,g^{\nu}(A_i)\right]\otimes\left[f^{\nu}(B_j)\,g^{1-\nu}(B_j)\right] \tag{3.3}$$
$$\leq (1-\nu)\sum_{i,j=1}^{n} p_ip_j f(A_i)\otimes g(B_j)+\nu\sum_{i,j=1}^{n} p_ip_j g(A_i)\otimes f(B_j)$$
$$\leq U^R(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2)\sum_{i,j=1}^{n} p_ip_j\left[f^{1-\nu}(A_i)\,g^{\nu}(A_i)\right]\otimes\left[f^{\nu}(B_j)\,g^{1-\nu}(B_j)\right],$$

which is equivalent to (3.1). ∎

Remark 2. Assume that f, g are continuous on I and

$$0<\gamma\leq\frac{f(t)}{g(t)}\leq\Gamma \qquad \text{for } t\in I.$$

For $B_i = A_i$, $i\in\{1,\ldots,n\}$ we get from (3.1) that

$$\left[\sum_{i=1}^{n} p_i f^{1-\nu}(A_i)g^{\nu}(A_i)\right]\otimes\left[\sum_{i=1}^{n} p_i f^{\nu}(A_i)g^{1-\nu}(A_i)\right]$$
$$\leq (1-\nu)\left(\sum_{i=1}^{n} p_i f(A_i)\right)\otimes\left(\sum_{i=1}^{n} p_i g(A_i)\right) \tag{3.4}$$
$$+\,\nu\left(\sum_{i=1}^{n} p_i g(A_i)\right)\otimes\left(\sum_{i=1}^{n} p_i f(A_i)\right)$$
$$\leq\left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R\left[\sum_{i=1}^{n} p_i f^{1-\nu}(A_i)g^{\nu}(A_i)\right]\otimes\left[\sum_{i=1}^{n} p_i f^{\nu}(A_i)g^{1-\nu}(A_i)\right].$$

From (3.4) we get a similar inequality for the Hadamard product "$\circ$".

4. Examples

Assume that the operators A and B satisfy the conditions

$$0<m\leq A,B\leq M$$

for some constants m and M.

Consider the functions $f(t) = t^p$, $g(t) = t^q$ for $t > 0$ and $p \neq q$ are real numbers. We have $\frac{t^p}{t^q} = t^{p-q}$ and

$$m^{p-q} \leq \frac{f(t)}{g(t)} \leq M^{p-q} \qquad \text{for } p > q$$

and

$$M^{p-q} \leq \frac{f(t)}{g(t)} \leq m^{p-q} \qquad \text{for } p < q$$

for all $t \in [m, M]$.

For $p > q$ we get by Corollary 2 that

$$\begin{aligned} A^{(1-\nu)p+\nu q} \otimes B^{\nu p+(1-\nu)q} &\leq (1-\nu) A^p \otimes B^q + \nu A^q \otimes B^p \\ &\leq \left[\frac{(m^{p-q} + M^{p-q})^2}{4m^{p-q}M^{p-q}}\right]^R A^{(1-\nu)p+\nu q} \otimes B^{\nu p+(1-\nu)q} \end{aligned} \tag{4.1}$$

where $\nu \in [0,1]$ and $R = \max\{1-\nu, \nu\}$.

In particular,

$$\begin{aligned} A^{\frac{p+q}{2}} \otimes B^{\frac{p+q}{2}} &\leq \frac{1}{2}[A^p \otimes B^q + A^q \otimes B^p] \\ &\leq \frac{m^{p-q} + M^{p-q}}{2m^{\frac{p-q}{2}} M^{\frac{p-q}{2}}} A^{\frac{p+q}{2}} \otimes B^{\frac{p+q}{2}}. \end{aligned} \tag{4.2}$$

We also have for $B = A$ that

$$\begin{aligned} A^{(1-\nu)p+\nu q} \otimes A^{\nu p+(1-\nu)q} &\leq (1-\nu) A^p \otimes A^q + \nu A^q \otimes A^p \\ &\leq \left[\frac{(m^{p-q} + M^{p-q})^2}{4m^{p-q}M^{p-q}}\right]^R A^{(1-\nu)p+\nu q} \otimes A^{\nu p+(1-\nu)q}. \end{aligned} \tag{4.3}$$

In particular,

$$\begin{aligned} A^{\frac{p+q}{2}} \otimes A^{\frac{p+q}{2}} &\leq \frac{1}{2}[A^p \otimes A^q + A^q \otimes A^p] \\ &\leq \frac{m^{p-q} + M^{p-q}}{2m^{\frac{p-q}{2}} M^{\frac{p-q}{2}}} A^{\frac{p+q}{2}} \otimes A^{\frac{p+q}{2}}. \end{aligned} \tag{4.4}$$

The above inequalities (4.1)-(4.1) also hold for the Hadamard product "$\circ$" instead of tensorial product "$\otimes$".

Similar inequalities may be stated if one consider the functions $f(t) = \exp(\alpha t)$, $g(t) = \exp(\beta t)$ with $\alpha \neq \beta$ and $t \in \mathbb{R}$. The details are omitted.

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