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Abstract: Consider the power series with complex coefficients $h(z) = \sum_{k=0}^{\infty} a_k z^k$ and its modified version $h_a(z) = \sum_{k=0}^{\infty} |a_k| z^k$. In this article, we explore the application of certain Hölder-type inequalities for deriving various inequalities for operators acting on the aforementioned power series. We establish these inequalities under the assumption of the convergence of $h(z)$ on the open disk $D(0, \rho)$, where ρ denotes the radius of convergence. Additionally, we investigate the norm and numerical radius inequalities associated with these concepts.

Keywords: Hölder-type inequalities; power series; operators; operator norm; Hilbert spaces; numerical radius

MSC: 30B10; 47A30; 47B65; 47A13; 47A12

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1. Introduction and Preliminary

In mathematics, inequalities are fundamental tools for comparing and analyzing mathematical objects. This article focuses on a specific type of inequality called Hölder-type inequalities, which are applied to power series of operators in Hilbert spaces. This topic is important in the fields of operator theory and functional analysis. Our goal is to enhance the theoretical foundations of mathematical inequalities and contribute to the overall understanding of this subject within the mathematical community. Our research represents a significant advancement in this area, providing new insights and tools for mathematicians working in these fields. Inequalities are essential for establishing the properties of operators and investigating the convergence and behavior of power series. For further reading on mathematical inequalities, interested readers can consult recent papers and the references therein [1–9].

Consider the power series $h(z) = \sum_{k=0}^{\infty} a_k z^k$, where a_k represents complex numbers and z denotes a complex variable. Let us assume that the convergence of $h(z)$ occurs within a specific region, known as the open disk $D(0, \rho)$. This region comprises all complex numbers z with a distance less than ρ from the origin. If ρ is infinite, it signifies the convergence of the power series for all complex numbers.

Associated with the power series $h(z) = \sum_{k=0}^{\infty} a_k z^k$ is another series, denoted as $h_a(z) = \sum_{k=0}^{\infty} |a_k| z^k$. In this series, the coefficients are obtained by taking the absolute values of the coefficients in the original series $h(z)$. Both $h(z)$ and $h_a(z)$ share the same radius of convergence. One noteworthy case is when all coefficients a_k are non-negative, meaning $a_k \geq 0$ for all k. In this situation, the series $h_a(z)$ is equal to $h(z)$. Power series of operators are fundamental in functional analysis and operator theory, offering a systematic way to express and investigate operators. By using power series, one can explore operator properties and behavior in a structured manner. Readers interested in a deeper understanding of this topic can refer to references such as [10–13], which provide comprehensive insights into the power series of operators and their applications.

To illustrate the concepts mentioned earlier, consider some natural examples of power series:

$$h(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = \ln\left(\frac{1}{1+z}\right), \quad z \in D(0,1);$$
$$h(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \quad z \in \mathbb{C};$$
$$h(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \quad z \in D(0,1)$$

then, corresponding functions with absolute values of coefficients are then provided by:

$$h_a(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln\left(\frac{1}{1-z}\right), \quad z \in D(0,1);$$
$$h_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \quad z \in \mathbb{C};$$
$$h_a(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad z \in D(0,1).$$

Other notable examples of functions expressed as power series with nonnegative coefficients include:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad z \in \mathbb{C};$$
$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \quad z \in D(0,1).$$

Before delving into our exploration, it is crucial to revisit some fundamental definitions and concepts. Consider $\mathcal{B}(\mathcal{H})$, the C^*-algebra comprising all bounded linear operators on a complex Hilbert space $\mathcal{H}$. Let $T \in \mathcal{B}(\mathcal{H})$. The operator norm of T, denoted by $\|T\|$, is defined as the supremum of $\|Tx\|$ over all unit vectors $\|x\| = 1$, expressed as $\|T\| = \sup_{\|x\|=1} \|Tx\|$. In this context, for x in $\mathcal{H}$, the quantity $\|x\|$ is defined as the square root of the inner product $\langle x, x\rangle$, where $\langle \cdot, \cdot \rangle$ symbolizes the inner product on $\mathcal{H}$. Alternatively, the operator norm $\|\cdot\|$ can be defined as $\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|$. By setting $y = x$ in this definition, a smaller quantity emerges known as the numerical radius, denoted by $\omega(T)$. Thus, for $T \in \mathcal{B}(\mathcal{H})$, the numerical radius of T is the scalar value $\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. Importantly, $\omega(\cdot)$ also defines a norm on $\mathcal{B}(\mathcal{H})$. Nevertheless, noteworthy distinctions exist between the norm properties of $\omega(\cdot)$ and $\|\cdot\|$. Specifically, the numerical radius lacks sub-multiplicativity and unitary invariance, in contrast with the operator norm.

Even though understanding $\omega(\cdot)$ might seem simpler than $\|\cdot\|$, determining the numerical radius $\omega(\cdot)$ is actually more challenging. As a result, there has been significant interest in the research community in estimating the values of $\omega(\cdot)$ in terms of the operator norm $\|\cdot\|$. This is often achieved by establishing sharp upper and lower bounds. In this context, an important relationship, as discussed in (Theorem 1.3-1 [14]), states that for every $T \in \mathcal{B}(\mathcal{H})$, we have

$$\omega(T) \le \|T\| \le 2\omega(T) \tag{1}$$

This connection shows that the two norms, $\omega(\cdot)$ and $\|\cdot\|$, are related. However, it is crucial to understand that there might be a significant difference between the values on the left and right sides of (1). Consequently, researchers have dedicated considerable efforts to finding better bounds for more accurate approximations and a deeper understanding of these relationships. For more information on norm and numerical radius inequalities, readers are encouraged to consult the following references [15–26] and the additional references cited therein.

The paper is structured as follows. In Section 2, our main focus is on establishing various vector inequalities for operators. We delve into the summation of the power series of operators in Hilbert spaces and their modified versions. We also provide several generalizations of a Kato-type inequality for Hölder weighted sums of operators, as established in [27]. Among other results, we demonstrate that if the power series with complex coefficients $h(\lambda) := \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $D(0,\rho)$ and X_i, U_i, $V_i \in \mathcal{B}(\mathcal{H})$ with $\|X_i\| < \rho$, $i \in \{1,...,n\}$, then, for non-negative weights $p_i \geq 0$ with $\sum_{i=1}^{n} p_i > 0$ (meaning that not all of them are zero), it holds that:

$$\left|\left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, y\right\rangle\right|^2 \leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \left\langle \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \big||X_i|^{\alpha} U_i\big|^{2p} x, x\right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \Big||X_i^*|^{1-\alpha} V_i\Big|^{2q} y, y\right\rangle^{\frac{1}{q}}.$$

for all $x, y \in \mathcal{H}$, $\alpha \in [0,1]$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

If the power series h reduces to the constant 1, then we obtain the usual Hölder's-type vector inequality for weighted sums

$$\left|\left\langle \sum_{i=1}^{n} p_i V_i^* X_i U_i x, y\right\rangle\right|^2 \leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \left\langle \sum_{i=1}^{n} p_i \big||X_i|^{\alpha} U_i\big|^{2p} x, x\right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^{n} p_i \Big||X_i^*|^{1-\alpha} V_i\Big|^{2q} y, y\right\rangle^{\frac{1}{q}}.$$

When $V_i = U_i = I$ for all $i \in \{1,...,n\}$, we obtain the one sequence vector inequality for weighted sums

$$\left|\left\langle \sum_{i=1}^{n} p_i X_i x, y\right\rangle\right|^2 \leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \left\langle \sum_{i=1}^{n} p_i |X_i|^{2\alpha p} x, x\right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^{n} p_i |X_i^*|^{2(1-\alpha)q} y, y\right\rangle^{\frac{1}{q}}. \tag{2}$$

Moreover, for $n = 2$ and $p_1 = p_2 = 1$, we derive from (2) the following Hölder type vector inequality for the sum of two operators

$$|\langle (A+B)x, y\rangle|^2 \leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \left\langle \left(|A|^{2\alpha p} + |B|^{2\alpha p}\right) x, x\right\rangle^{\frac{1}{p}} \left\langle \left(|A^*|^{2(1-\alpha)q} + |B^*|^{2(1-\alpha)q}\right) y, y\right\rangle^{\frac{1}{q}}$$

for $A, B \in \mathcal{B}(\mathcal{H})$, $x, y \in \mathcal{H}$, $\alpha \in [0,1]$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

In Section 3, we discuss a range of inequalities related to the norm and numerical radius. As an example, we highlight the following result: if $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $s\geq\max\{p,q\}>1$, then

$$\begin{aligned}&\omega^{2s}\left(\sum_{i=1}^{n}p_iV_i^*X_ih(X_i)U_i\right)\\&\leq\frac{1}{2}\left\|\sum_{i=1}^{n}p_ih_a^p(\|X_i\|)\left||X_i|^{\alpha}U_i\right|^{2p}\right\|^{\frac{s}{p}}\left\|\sum_{i=1}^{n}p_ih_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha}V_i\right|^{2q}\right\|^{\frac{s}{q}}\\&+\frac{1}{2}\omega\left[\left(\sum_{i=1}^{n}p_ih_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha}V_i\right|^{2q}\right)^{\frac{s}{q}}\left(\sum_{i=1}^{n}p_ih_a^p(\|X_i\|)\left||X_i|^{\alpha}U_i\right|^{2p}\right)^{\frac{s}{p}}\right]\end{aligned}$$

provided that the power series with complex coefficients $h(\lambda):=\sum_{k=0}^{\infty}a_k\lambda^k$ is convergent on $D(0,\rho)$, X_i, U_i, $V_i\in\mathcal{B}(\mathcal{H})$ with $\|X_i\|<\rho$, $i\in\{1,...,n\}$, $\alpha\in[0,1]$ and $p_i\geq 0$ with $\sum_{i=1}^{n}p_i>0$.

Here, if $h\equiv 1$, then the above result becomes the norm and numerical radius inequality for weighted sums:

$$\begin{aligned}&\omega^{2s}\left(\sum_{i=1}^{n}p_iV_i^*X_iU_i\right)\\&\leq\frac{1}{2}\left\|\sum_{i=1}^{n}p_i\left||X_i|^{\alpha}U_i\right|^{2p}\right\|^{\frac{s}{p}}\left\|\sum_{i=1}^{n}p_i\left||X_i^*|^{1-\alpha}V_i\right|^{2q}\right\|^{\frac{s}{q}}\\&+\frac{1}{2}\omega\left[\left(\sum_{i=1}^{n}p_i\left||X_i^*|^{1-\alpha}V_i\right|^{2q}\right)^{\frac{s}{q}}\left(\sum_{i=1}^{n}p_i\left||X_i|^{\alpha}U_i\right|^{2p}\right)^{\frac{s}{p}}\right].\end{aligned}$$

In particular, $V_i=U_i=I$ for all $i\in\{1,...,n\}$, we obtain the one sequence numerical radius inequality for weighted sums

$$\begin{aligned}&\omega^{2s}\left(\sum_{i=1}^{n}p_iX_i\right)\\&\leq\frac{1}{2}\left\|\sum_{i=1}^{n}p_i|X_i|^{2\alpha p}\right\|^{\frac{s}{p}}\left\|\sum_{i=1}^{n}p_i|X_i^*|^{2(1-\alpha)q}\right\|^{\frac{s}{q}}\\&+\frac{1}{2}\omega\left[\left(\sum_{i=1}^{n}p_i|X_i^*|^{2(1-\alpha)q}\right)^{\frac{s}{q}}\left(\sum_{i=1}^{n}p_i|X_i|^{2(\alpha)p}\right)^{\frac{s}{p}}\right].\end{aligned}\tag{3}$$

Moreover, for $n=2$ and $p_1=p_2=1$,, we derive from (3) the following Hölder-type numerical radius inequality for the sum of two operators

$$\begin{aligned}\omega^{2s}(A+B)&\leq\frac{1}{2}\left\||A|^{2\alpha p}+|B|^{2\alpha p}\right\|^{\frac{s}{p}}\left\||A^*|^{2(1-\alpha)q}+|B^*|^{2(1-\alpha)q}\right\|^{\frac{s}{q}}\\&+\frac{1}{2}\omega\left[\left(|A^*|^{2(1-\alpha)q}+|B^*|^{2(1-\alpha)q}\right)^{\frac{s}{q}}\left(|A|^{2\alpha p}+|B|^{2\alpha p}\right)^{\frac{s}{p}}\right]\end{aligned}\tag{4}$$

for A, $B\in\mathcal{B}(\mathcal{H})$, $\alpha\in[0,1]$ and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $s\geq\max\{p,q\}>1$.

For $p = q = 2$, we obtain from (4) that

$$\omega^{2s}(A+B) \le \frac{1}{2}\left\| |A|^{4\alpha} + |B|^{4\alpha} \right\|^{\frac{s}{2}} \left\| |A^*|^{4(1-\alpha)} + |B^*|^{4(1-\alpha)} \right\|^{\frac{s}{2}} + \frac{1}{2}\omega\left[\left(|A^*|^{4(1-\alpha)} + |B^*|^{4(1-\alpha)} \right)^{\frac{s}{2}} \left(|A|^{4\alpha} + |B|^{4\alpha} \right)^{\frac{s}{2}} \right]$$

for $\alpha \in [0,1]$ and $s \ge 2$, which for $\alpha = 1/2$ provides

$$\omega^{2s}(A+B) \le \frac{1}{2}\left\| |A|^{2} + |B|^{2} \right\|^{\frac{s}{2}} \left\| |A^*|^{2} + |B^*|^{2} \right\|^{\frac{s}{2}} + \frac{1}{2}\omega\left[\left(|A^*|^{2} + |B^*|^{2} \right)^{\frac{s}{2}} \left(|A|^{2} + |B|^{2} \right)^{\frac{s}{2}} \right]$$

for $s \ge 2$. Finally, if we take $s = 2$, we also receive

$$\omega^{4}(A+B) \le \frac{1}{2}\left\| |A|^{2} + |B|^{2} \right\| \left\| |A^*|^{2} + |B^*|^{2} \right\| + \frac{1}{2}\omega\left[\left(|A^*|^{2} + |B^*|^{2} \right) \left(|A|^{2} + |B|^{2} \right) \right] \tag{5}$$

for A, $B \in \mathcal{B}(\mathcal{H})$.

We observe that the above inequalities (3)–(5) provide some complementary results for the numerical radius inequalities for the finite sums obtained recently in [28,29]. As far as we can see, the upper bounds for the numerical radius obtained in this paper cannot be compared with any bound from the papers [28,29].

To illustrate our theoretical results, we provide various examples of fundamental operator functions such as the resolvent, the logarithm function, operator exponential, and operator trigonometric and hyperbolic functions.

2. Vector Inequalities Involving Power Series of Operators

In order to establish our initial result in this section, it is necessary to invoke the following vector inequality for positive operators $A \ge 0$, as derived by McCarthy in [30]:

$$\langle Ax, x \rangle^p \le \langle A^p x, x \rangle, \quad p \ge 1,$$

where $x \in \mathcal{H}$ and $\|x\| = 1$. Additionally, we utilize Buzano's inequality [31]:

$$|\langle x, e \rangle \langle e, y \rangle| \le \frac{1}{2}[\|x\|\|y\| + |\langle x, y \rangle|], \tag{6}$$

which holds for any $x, y, e \in \mathcal{H}$ with $\|e\| = 1$.

Substituting x with $\frac{y}{\|y\|}$, where $y \ne 0$, into (6), we obtain

$$\langle A\frac{y}{\|y\|}, \frac{y}{\|y\|} \rangle^p \le \langle A^p \frac{y}{\|y\|}, \frac{y}{\|y\|} \rangle, \quad p \ge 1,$$

which can be expressed as

$$\langle Ay, y \rangle^p \le \|y\|^{2(p-1)} \langle A^p y, y \rangle, \quad p \ge 1, \tag{7}$$

valid for all $y \in \mathcal{H}$.

In this section, we consider the power series with complex coefficients $h(\lambda) := \sum_{k=0}^{\infty} a_k \lambda^k$ with $a_k \in \mathbb{C}$ for $k \in \mathbb{N} := \{0, 1, \ldots\}$. We assume that this power series is convergent on the open disk $D(0,\rho) := \{z \in \mathbb{C}; |z| < \rho\}$. If $\rho = \infty$, then $D(0,\rho) = \mathbb{C}$. We define $h_a(\lambda) := \sum_{k=0}^{\infty} |a_k| \lambda^k$, which has the same radius of convergence ρ.

To prove our first result, we need to establish the following lemma concerning a generalized version of Schwarz vector inequlity concerning the natural powers of an operator T from $\mathcal{B}(\mathcal{H})$.

Lemma 1. *Let $T, U, V \in \mathcal{B}(\mathcal{H})$ and $\alpha \in [0,1]$. Then, for $n \geq 1$ we have*

$$\left|\langle V^* T^n U x, y\rangle\right|^2 \leq \|T\|^{2n-2} \left\langle \left||T|^\alpha U\right|^2 x, x\right\rangle \left\langle \left||T^*|^{1-\alpha} V\right|^2 y, y\right\rangle \tag{8}$$

for all $x, y \in \mathcal{H}$.

Proof. Firstly, observe that Kittaneh derived the following Schwarz-type inequality for powers of operators in [32]. This inequality asserts that for every $T \in \mathcal{B}(\mathcal{H})$, and for all $x, y \in \mathcal{H}$, $\alpha \in [0,1]$ and $n \geq 1$, the following holds:

$$\left|\langle T^n x, y\rangle\right|^2 \leq \|T\|^{2n-2} \langle |T|^{2\alpha} x, x\rangle \langle |T^*|^{2(1-\alpha)} y, y\rangle. \tag{9}$$

Now, let $x, y \in \mathcal{H}$. If we replace x by Ux and y by Vy in (9), then we get

$$\left|\langle V^* T^n U x, y\rangle\right|^2 \leq \|T\|^{2n-2} \langle U^* |T|^{2\alpha} U x, x\rangle \langle V^* |T^*|^{2(1-\alpha)} V y, y\rangle. \tag{10}$$

Observe that $U^*|T|^{2\alpha} U = \left||T|^\alpha U\right|^2$ and $V^*|T^*|^{2(1-\alpha)} V = \left||T^*|^{1-\alpha} V\right|^2$, then by (10), we get (8). □

Now, we are able to establish the following result.

Proposition 1. *Assume that the power series with complex coefficients $h(\lambda) := \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $D(0, \rho)$ and $T, U, V \in \mathcal{B}(\mathcal{H})$ with $\|T\| < \rho$, then*

$$\left|\langle V^* T h(T) U x, y\rangle\right|^2 \leq h_a^2(\|T\|) \left\langle \left||T|^\alpha U\right|^2 x, x\right\rangle \left\langle \left||T^*|^{1-\alpha} V\right|^2 y, y\right\rangle \tag{11}$$

for $\alpha \in [0,1]$ and $x, y \in \mathcal{H}$. In particular,

$$\left|\langle V^* T h(T) U x, y\rangle\right|^2 \leq h_a^2(\|T\|) \left\langle \left||T|^{\frac{1}{2}} U\right|^2 x, x\right\rangle \left\langle \left||T^*|^{\frac{1}{2}} V\right|^2 y, y\right\rangle \tag{12}$$

for $x, y \in \mathcal{H}$.

Proof. If we take $n = k+1$, $k \in \mathbb{N}$ in (8) and take the square root, then we obtain

$$\left|\langle V^* T T^k U x, y\rangle\right| \leq \|T\|^k \left\langle \left||T|^\alpha U\right|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle \left||T^*|^{1-\alpha} V\right|^2 y, y\right\rangle^{\frac{1}{2}}$$

for all $x, y \in \mathcal{H}$.

Furthermore, if we multiply by $|a_k| \geq 0$, where $k \in \{0, 1, ...\}$, and sum over k from 0 to m, then we obtain

$$\begin{aligned}
\left|\left\langle V^* T \sum_{k=0}^{m} a_k T^k U x, y\right\rangle\right| &= \left|\sum_{k=0}^{m} a_k \langle V^* T T^k U x, y\rangle\right| \\
&\leq \sum_{k=0}^{m} |a_k| \left|\langle V^* T T^k U x, y\rangle\right| \\
&\leq \sum_{k=0}^{m} |a_k| \|T\|^k \left\langle \left||T|^\alpha U\right|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle \left||T^*|^{1-\alpha} V\right|^2 y, y\right\rangle^{\frac{1}{2}}
\end{aligned} \tag{13}$$

for all $x, y \in \mathcal{H}$.

As $\|T\| < \rho_{,}$, then series $\sum_{k=0}^{\infty} a_k T^k$ and $\sum_{k=0}^{\infty} |a_k| \|T\|^k$ are convergent and

$$\sum_{k=0}^{\infty} a_k T^k = h(T) \text{ and } \sum_{k=0}^{\infty} |a_k| \|T\|^k = h_a(\|T\|).$$

By taking the limit over $m \to \infty$ in (13), we deduce the desired result (11). □

The following two remarks are crucial as they reveal significant consequences derived from the preceding proposition. These remarks provide valuable insights into the broader implications of the results obtained, further enhancing our understanding of the theoretical framework.

Remark 1. *(1) If we take $h \equiv 1$, in (11) and (12), then we obtain the following Kato-type inequality [32]*

$$\left|\langle V^* T U x, y\rangle\right|^2 \le \left\langle \left||T|^{\alpha} U\right|^2 x, x\right\rangle \left\langle \left||T^*|^{1-\alpha} V\right|^2 y, y\right\rangle$$

for $\alpha \in [0,1]$ and $x, y \in \mathcal{H}$. In particular,

$$\left|\langle V^* T U x, y\rangle\right|^2 \le \left\langle \left||T|^{\frac{1}{2}} U\right|^2 x, x\right\rangle \left\langle \left||T^*|^{\frac{1}{2}} V\right|^2 y, y\right\rangle.$$

(2) If we take $U = V = I$ in (11) and (12), then we obtain for $\alpha \in [0,1]$ that

$$\left|\langle T h(T) x, y\rangle\right|^2 \le h_a^2(\|T\|) \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle \tag{14}$$

and

$$\left|\langle T h(T) x, y\rangle\right|^2 \le h_a^2(\|T\|) \langle |T| x, x\rangle \langle |T^*| y, y\rangle$$

for $x, y \in \mathcal{H}$.

The case $h \equiv 1$ provides the original Kato's inequality [32]; therefore, (14) can be seen as a functional extension of Kato's celebrated result in the case when the function is provided by a power series.

(3) If T is invertible and we take $V = I$, $U = T^{-1}$ in (11), then we obtain

$$\left|\langle h(T) x, y\rangle\right|^2 \le h_a^2(\|T\|) \left\langle \left||T|^{\alpha} T^{-1}\right|^2 x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle$$

for $\alpha \in [0,1]$ and $x, y \in \mathcal{H}$. In particular,

$$\left|\langle h(T) x, y\rangle\right|^2 \le h_a^2(\|T\|) \left\langle \left||T|^{\frac{1}{2}} T^{-1}\right|^2 x, x\right\rangle \langle |T^*| y, y\rangle$$

for $x, y \in \mathcal{H}$.

(4) If $T > 0$ and we take $U = T^{-\beta}$, $V = T^{-1+\beta}$, $\beta \in [0,1]$, then we derive

$$\left|\langle h(T) x, y\rangle\right|^2 \le h_a^2(\|T\|) \left\langle T^{2(\alpha-\beta)} x, x\right\rangle \left\langle T^{2(\beta-\alpha)} y, y\right\rangle$$

for $\alpha \in [0,1]$ and $x, y \in \mathcal{H}$.

To further clarify the previous result, we provide helpful examples in the following remarks. This will aid in understanding the concepts and implications presented earlier for some fundamental operator functions.

Remark 2. *If $T, U, V \in \mathcal{B}(\mathcal{H})$ with $\|T\| < 1$, then for $\alpha \in [0,1]$ we have the following inequalities involving the resolvent functions $(I \pm T)^{-1}$*

$$\left|\left\langle V^* T (I \pm T)^{-1} U x, y\right\rangle\right|^2 \le (1 - \|T\|)^{-2} \left\langle \left||T|^{\alpha} U\right|^2 x, x\right\rangle \left\langle \left||T^*|^{1-\alpha} V\right|^2 y, y\right\rangle \tag{15}$$

and inequalities involving the operator entropy functions $T\ln(I \pm T)$

$$\left|\left\langle V^*T\ln(I \pm T)Ux, y\right\rangle\right|^2 \le [\ln(1-\|T\|)]^2\left\langle \left||T|^\alpha U\right|^2 x, x\right\rangle\left\langle \left||T^*|^{1-\alpha}V\right|^2 y, y\right\rangle \tag{16}$$

for all $x, y \in \mathcal{H}$.

Remark 3. *For* $\alpha = \frac{1}{2}$ *in (15) and (16), we obtain*

$$\left|\left\langle V^*T(I \pm T)^{-1}Ux, y\right\rangle\right|^2 \le (1-\|T\|)^{-2}\langle U^*|T|Ux, x\rangle\langle V^*|T^*|Vy, y\rangle$$

and

$$\left|\left\langle V^*T\ln(I \pm T)Ux, y\right\rangle\right|^2 \le [\ln(1-\|T\|)]^2\langle U^*|T|Ux, x\rangle\langle V^*|T^*|Vy, y\rangle$$

for all $x, y \in \mathcal{H}$.

Remark 4. *If* $T, U, V \in \mathcal{B}(\mathcal{H})$ *and* $\alpha \in [0,1]$, *then we have the following results connecting the operator trigonometric and hyperbolic functions can be stated as well*

$$\left|\left\langle V^*T\sin(T)Ux, y\right\rangle\right|^2 \le [\sinh(\|T\|)]^2\left\langle \left||T|^\alpha U\right|^2 x, x\right\rangle\left\langle \left||T^*|^{1-\alpha}V\right|^2 y, y\right\rangle \tag{17}$$

and

$$\left|\left\langle V^*T\cos(T)Ux, y\right\rangle\right|^2 \le [\cosh(\|T\|)]^2\left\langle \left||T|^\alpha U\right|^2 x, x\right\rangle\left\langle \left||T^*|^{1-\alpha}V\right|^2 y, y\right\rangle \tag{18}$$

for all $x, y \in \mathcal{H}$.

Remark 5. *For* $\alpha = \frac{1}{2}$ *in (17) and (18) we obtain*

$$\left|\left\langle V^*T\sin(T)Ux, y\right\rangle\right|^2 \le [\sinh(\|T\|)]^2\langle U^*|T|Ux, x\rangle\langle V^*|T^*|Vy, y\rangle$$

and

$$\left|\left\langle V^*T\cos(T)Ux, y\right\rangle\right|^2 \le [\cosh(\|T\|)]^2\langle U^*|T|Ux, x\rangle\langle V^*|T^*|Vy, y\rangle$$

for all $x, y \in \mathcal{H}$.

Remark 6. *Also, if* $T, U, V \in \mathcal{B}(\mathcal{H})$ *and* $\alpha \in [0,1]$, *then we have the following results involving the operator exponential and the hyperbolic functions*

$$\left|\left\langle V^*T\exp(T)Ux, y\right\rangle\right|^2 \le \exp(2\|T\|)\left\langle \left||T|^\alpha U\right|^2 x, x\right\rangle\left\langle \left||T^*|^{1-\alpha}V\right|^2 y, y\right\rangle,$$

$$\left|\left\langle V^*T\sinh(T)Ux, y\right\rangle\right|^2 \le [\sinh(\|T\|)]^2\left\langle \left||T|^\alpha U\right|^2 x, x\right\rangle\left\langle \left||T^*|^{1-\alpha}V\right|^2 y, y\right\rangle$$

and

$$\left|\left\langle V^*T\cosh(T)Ux, y\right\rangle\right|^2 \le [\cosh(\|T\|)]^2\left\langle \left||T|^\alpha U\right|^2 x, x\right\rangle\left\langle \left||T^*|^{1-\alpha}V\right|^2 y, y\right\rangle$$

for all $x, y \in \mathcal{H}$.

Remark 7. *For* $\alpha = \frac{1}{2}$ *in the last three equations, we obtain some simpler inequalities. However, we omit the details.*

Our next result provides another important finding involving vector inequalities for a power series of operators. It reads as follows:

Theorem 1. *Let* $h(z) := \sum_{k=0}^{\infty} a_k z^k$ *be a convergent power series with complex coefficients on* $D(0,\rho)$. *Take* X_i, U_i, $V_i \in \mathcal{L}(\mathbb{H})$ *with* $\|X_i\| < \rho$ *for* $i \in \{1,\ldots,n\}$. *Choose* $p, q > 1$ *such that*

$\frac{1}{p} + \frac{1}{q} = 1$. *Then, for non-negative* p_i *(*$i = 1$ *to* n*) with* $\sum_{i=1}^{n} p_i > 0$*, the following inequalities hold for all* $x, y \in \mathbb{H}$ *and* $\alpha \in [0,1]$*:*

$$\begin{aligned} &\left| \left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, y \right\rangle \right|^2 \\ &\le \sum_{i=1}^{n} p_i \left| \left\langle V_i^* X_i h(X_i) U_i x, y \right\rangle \right|^2 \\ &\le \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \left\langle \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left| |X_i|^\alpha U_i \right|^{2p} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} y, y \right\rangle^{\frac{1}{q}}. \end{aligned} \tag{19}$$

Proof. From (11) we have

$$\left| \left\langle V_i^* X_i h(X_i) U_i x, y \right\rangle \right|^2 \le \left\langle h_a(\|X_i\|) \left| |X_i|^\alpha U_i \right|^2 x, x \right\rangle \left\langle h_a(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle \tag{20}$$

for all $x, y \in \mathcal{H}$ and $i \in \{1, ..., n\}$.

If we multiply (20) by $p_i \ge 0$, $i \in \{1, ..., n\}$ and sum over i from 1 to n, then we obtain

$$\sum_{i=1}^{n} p_i \left| \left\langle V_i^* X_i h(X_i) U_i x, y \right\rangle \right|^2 \le \sum_{i=1}^{n} p_i \left\langle h_a(\|X_i\|) \left| |X_i|^\alpha U_i \right|^2 x, x \right\rangle \left\langle h_a(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle$$

for all $x, y \in \mathcal{H}$.

From the Cauchy–Buniakowsky-Schwarz weighted inequality we have

$$\begin{aligned} \left| \left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, y \right\rangle \right|^2 &= \left| \sum_{i=1}^{n} p_i \left\langle V_i^* X_i h(X_i) U_i x, y \right\rangle \right|^2 \\ &\le \sum_{i=1}^{n} p_i \left| \left\langle V_i^* X_i h(X_i) U_i x, y \right\rangle \right|^2 \end{aligned} \tag{21}$$

for all $x, y \in \mathcal{H}$.

From weighted Hölder's inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} &\sum_{i=1}^{n} p_i \left\langle h_a(\|X_i\|) \left| |X_i|^\alpha U_i \right|^2 x, x \right\rangle \left\langle h_a(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle \\ &\le \left(\sum_{i=1}^{n} p_i \left\langle h_a(\|X_i\|) \left| |X_i|^\alpha U_i \right|^2 x, x \right\rangle^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} p_i \left\langle h_a(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle^q \right)^{\frac{1}{q}} \end{aligned} \tag{22}$$

for all $x, y \in \mathcal{H}$.

From the McCarthy inequality (7) we have

$$\left\langle h_a(\|X_i\|) \left| |X_i|^\alpha U_i \right|^2 x, x \right\rangle^p \le \|x\|^{2(p-1)} h_a^p(\|X_i\|) \left\langle \left| |X_i|^\alpha U_i \right|^{2p} x, x \right\rangle$$

and

$$\left\langle h_a(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle^q \le \|y\|^{2(q-1)} h_a^q(\|X_i\|) \left\langle \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} y, y \right\rangle$$

for all $x, y \in \mathcal{H}$.

Therefore, from (22) we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} p_i \langle h_a(\|X_i\|) \big| |X_i|^\alpha U_i \big|^2 x, x \rangle \langle h_a(\|X_i\|) \Big| |X_i^*|^{1-\alpha} V_i \Big|^2 y, y \rangle \\
&\le \left(\|x\|^{2(p-1)} \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \langle \big| |X_i|^\alpha U_i \big|^{2p} x, x \rangle \right)^{\frac{1}{p}} \\
&\times \left(\|y\|^{2(q-1)} \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \langle \Big| |X_i^*|^{1-\alpha} V_i \Big|^{2q} y, y \rangle \right)^{\frac{1}{q}} \\
&= \|x\|^{2\left(1-\frac{1}{p}\right)} \left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \langle \big| |X_i|^\alpha U_i \big|^{2p} x, x \rangle \right)^{\frac{1}{p}} \\
&\times \|y\|^{2\left(1-\frac{1}{q}\right)} \left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \langle \Big| |X_i^*|^{1-\alpha} V_i \Big|^{2q} y, y \rangle \right)^{\frac{1}{q}} \\
&= \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \langle \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \big| |X_i|^\alpha U_i \big|^{2p} x, x \rangle^{\frac{1}{p}} \langle \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \Big| |X_i^*|^{1-\alpha} V_i \Big|^{2q} y, y \rangle^{\frac{1}{q}}.
\end{aligned} \tag{23}$$

By making use of (21)–(23), we obtain (19). □

Remark 8. *By letting $\alpha = \frac{1}{2}$ in Theorem 1, we deduce that*

$$\begin{aligned}
&\left| \langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, y \rangle \right|^2 \\
&\le \sum_{i=1}^{n} p_i \big| \langle V_i^* X_i h(X_i) U_i x, y \rangle \big|^2 \\
&\le \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \langle \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \Big| |X_i|^{\frac{1}{2}} U_i \Big|^{2p} x, x \rangle^{\frac{1}{p}} \langle \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \Big| |X_i^*|^{\frac{1}{2}} V_i \Big|^{2q} y, y \rangle^{\frac{1}{q}}
\end{aligned}$$

for all $x, y \in \mathcal{H}$.

Corollary 1. *With the assumptions of Theorem 1, we have*

$$\begin{aligned}
&\left| \langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, y \rangle \right|^4 \\
&\le \left(\sum_{i=1}^{n} p_i \big| \langle V_i^* X_i h(X_i) U_i x, y \rangle \big|^2 \right)^2 \\
&\le \|x\|^2 \|y\|^2 \langle \sum_{i=1}^{n} p_i h_a^2(\|X_i\|) \big| |X_i|^\alpha U_i \big|^4 x, x \rangle \langle \sum_{i=1}^{n} p_i h_a^2(\|X_i\|) \Big| |X_i^*|^{1-\alpha} V_i \Big|^4 y, y \rangle
\end{aligned}$$

for all $x, y \in \mathcal{H}$ and $\alpha \in [0, 1]$.

In particular,

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, y \right\rangle\right|^4 \\
&\leq \left(\sum_{i=1}^{n} p_i \left|\left\langle V_i^* X_i h(X_i) U_i x, y \right\rangle\right|^2\right)^2 \\
&\leq \|x\|^2 \|y\|^2 \left\langle \sum_{i=1}^{n} p_i h_a^2(\|X_i\|) \left| |X_i|^{\frac{1}{2}} U_i \right|^4 x, x \right\rangle \left\langle \sum_{i=1}^{n} p_i h_a^2(\|X_i\|) \left| |X_i^*|^{\frac{1}{2}} V_i \right|^4 y, y \right\rangle
\end{aligned}$$

for all $x, y \in \mathcal{H}$

Remark 9. *Since* $h_a(\cdot)$ *is a increasing function on* $(0, \rho)$*, then*

$$h_a(\|X_i\|) \leq h_a\left(\max_{k=1,\ldots,n} \|X_k\|\right) = \max_{k=1,\ldots,n} h_a(\|X_k\|),$$

then by (19) we derive for all $\alpha \in [0,1]$*,* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, that*

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, y \right\rangle\right|^2 \\
&\leq \sum_{i=1}^{n} p_i \left|\left\langle V_i^* X_i h(X_i) U_i x, y \right\rangle\right|^2 \\
&\leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \max_{k=1,\ldots,n} h_a(\|X_k\|) \left\langle \sum_{i=1}^{n} p_i \left| |X_i|^{\alpha} U_i \right|^{2p} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^{n} p_i \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} y, y \right\rangle^{\frac{1}{q}}
\end{aligned}$$

for all $x, y \in \mathcal{H}$ *and* $\alpha \in [0,1]$.

In particular, we have

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, y \right\rangle\right|^2 \\
&\leq \sum_{i=1}^{n} p_i \left|\left\langle V_i^* X_i h(X_i) U_i x, y \right\rangle\right|^2 \\
&\leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \max_{k=1,\ldots,n} h_a(\|X_k\|) \left\langle \sum_{i=1}^{n} p_i \left| |X_i|^{\frac{1}{2}} U_i \right|^{2p} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^{n} p_i \left| |X_i^*|^{\frac{1}{2}} V_i \right|^{2q} y, y \right\rangle^{\frac{1}{q}}
\end{aligned}$$

for all $x, y \in \mathcal{H}$.

Additional consequences arising from Theorem 1 are outlined in the following two remarks.

Remark 10. *If we take $V_i = U_i = I$ then for $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i > 0$, we obtain from Theorem 1 that*

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^n p_i X_i h(X_i) x, y \right\rangle\right|^2 \\
&\leq \sum_{i=1}^n p_i \left|\left\langle X_i h(X_i) x, y \right\rangle\right|^2 \\
&\leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \left\langle \sum_{i=1}^n p_i h_a^p(\|X_i\|) |X_i|^{2\alpha p} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^n p_i h_a^q(\|X_i\|) |X_i^*|^{2q(1-\alpha)} y, y \right\rangle^{\frac{1}{q}} \\
&\leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \max_{k=1,\ldots,n} h_a(\|X_k\|) \left\langle \sum_{i=1}^n p |X_i|^{2\alpha p} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^n p_i |X_i^*|^{2q(1-\alpha)} y, y \right\rangle^{\frac{1}{q}}
\end{aligned} \tag{24}$$

for all $x, y \in \mathbb{H}$ and $\alpha \in [0, 1]$. In particular, we have

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^n p_i X_i h(X_i) x, y \right\rangle\right|^2 \\
&\leq \sum_{i=1}^n p_i \left|\left\langle X_i h(X_i) x, y \right\rangle\right|^2 \\
&\leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \left\langle \sum_{i=1}^n p_i h_a^p(\|X_i\|) |X_i|^{p} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^n p_i h_a^q(\|X_i\|) |X_i^*|^{q} y, y \right\rangle^{\frac{1}{q}} \\
&\leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \max_{k=1,\ldots,n} h_a(\|X_k\|) \left\langle \sum_{i=1}^n p_i |X_i|^{p} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^n p_i |X_i^*|^{q} y, y \right\rangle^{\frac{1}{q}}
\end{aligned}$$

for all $x, y \in \mathbb{H}$.

Remark 11. *(1) If $X_i > 0$ and we take $U_i = X_i^{-\beta}$, $V_i = X_i^{-1+\beta}$, $\beta \in [0, 1]$, then we derive from Theorem 1 that*

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^n p_i h(X_i) x, y \right\rangle\right|^2 \\
&\leq \sum_{i=1}^n p_i \left|\left\langle h(X_i) x, y \right\rangle\right|^2 \\
&\leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \left\langle \sum_{i=1}^n p_i h_a^p(\|X_i\|) X_i^{2p(\alpha-\beta)} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^n p_i h_a^q(\|X_i\|) X_i^{2q(\beta-\alpha)} y, y \right\rangle^{\frac{1}{q}}
\end{aligned}$$

for all $x, y \in \mathcal{H}$ and $\alpha \in [0, 1]$.

(2) Now, if we take, for instance $h(\mu) = (1 \pm \mu)^{-1}$ with $|\mu| < 1$, then $h_a(\mu) = (1 - \mu)^{-1}$ and by (24) we get for all $x, y \in \mathcal{H}$ and $\alpha \in [0, 1]$ that

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^n p_i X_i (1 \pm X_i)^{-1} x, y \right\rangle\right|^2 \\
&\leq \sum_{i=1}^n p_i \left|\left\langle X_i (1 \pm X_i)^{-1} x, y \right\rangle\right|^2 \\
&\leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \left\langle \sum_{i=1}^n p_i (1 - \|X_i\|)^{-p} |X_i|^{2\alpha p} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^n p_i (1 - \|X_i\|)^{-q} |X_i^*|^{2q(1-\alpha)} y, y \right\rangle^{\frac{1}{q}},
\end{aligned}$$

where $\|X_i\| < 1$, $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$.

(3) Also, if we take $h(\mu) = \exp(c\mu)$ with $c, \mu \in \mathbb{C}$, then $h_a(\mu) = \exp(|c|\mu)$ and by (24) we get for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$ that

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n} p_i X_i \exp(cX_i)x, y\right\rangle\right|^2 \\
&\leq \sum_{i=1}^{n} p_i \left|\left\langle X_i \exp(cX_i)x, y\right\rangle\right|^2 \\
&\leq \|x\|^{\frac{2}{q}} \|y\|^{\frac{2}{p}} \left\langle \sum_{i=1}^{n} p_i \exp(p|c|\|X_i\|)|X_i|^{2\alpha p} x, x\right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^{n} p_i \exp(q|c|\|X_i\|)|X_i^*|^{2q(1-\alpha)} y, y\right\rangle^{\frac{1}{q}}
\end{aligned}$$

where $X_i \in \mathcal{B}(\mathcal{H})$, $p_i \geq 0$, $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^{n} p_i = 1$.

3. Norm and Numerical Radius Inequalities

In this section, our objective is to establish norm and numerical radius inequalities related to the power series $h(\cdot)$ and $h_a(\cdot)$. We begin by presenting our first result in this regard.

Theorem 2. *Let $h(z) := \sum_{k=0}^{\infty} a_k z^k$ be a convergent power series with complex coefficients on $D(0, \rho)$. Take X_i, U_i, $V_i \in \mathcal{B}(\mathcal{H})$ with $\|X_i\| < \rho$ for $i \in \{1, \ldots, n\}$. Choose $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for non-negative p_i ($i = 1$ to n) with $\sum_{i=1}^{n} p_i > 0$, the following inequalities hold for all $\alpha \in [0,1]$:*

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right\|^2 \qquad (25)\\
&\leq \left\|\sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^{\alpha} U_i\right|^{2p}\right\|^{\frac{1}{p}} \left\|\sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right\|^{\frac{1}{q}} \\
&\leq \max_{k=1,\ldots,n} h_a^2(\|X_k\|) \left\|\sum_{i=1}^{n} p_i \left||X_i|^{\alpha} U_i\right|^{2p}\right\|^{\frac{1}{p}} \left\|\sum_{i=1}^{n} p_i \left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right\|^{\frac{1}{q}}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\omega^2\left(\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right) \qquad (26)\\
&\leq \left\|\sum_{i=1}^{n} p_i \left(\frac{1}{p} h_a^p(\|X_i\|) \left||X_i|^{\alpha} U_i\right|^{2p} + \frac{1}{q} h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)\right\|.
\end{aligned}$$

Proof. From (19) we obtain

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right\|^2 \\
&= \sup_{\|x\|=\|y\|=1} \left|\left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, y\right\rangle\right|^2 \\
&\leq \sup_{\|x\|=1} \left\langle \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^{\alpha} U_i\right|^{2p} x, x\right\rangle^{\frac{1}{p}} \sup_{\|y\|=1} \left\langle \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q} y, y\right\rangle^{\frac{1}{q}} \\
&= \left\|\sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^{\alpha} U_i\right|^{2p}\right\|^{\frac{1}{p}} \left\|\sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right\|^{\frac{1}{q}},
\end{aligned}$$

which proves (25).

From Young's inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \; a, b \geq 0$$

we have that

$$\begin{aligned}
&\left\langle \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left| |X_i|^\alpha U_i \right|^{2p} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} x, x \right\rangle^{\frac{1}{q}} \qquad (27)\\
&\leq \frac{1}{p} \left\langle \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left| |X_i|^\alpha U_i \right|^{2p} x, x \right\rangle + \frac{1}{q} \left\langle \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} x, x \right\rangle \\
&= \left\langle \sum_{i=1}^{n} p_i \left(\frac{1}{p} h_a^p(\|X_i\|) \left| |X_i|^\alpha U_i \right|^{2p} + \frac{1}{q} h_a^q(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} \right) x, x \right\rangle
\end{aligned}$$

for $x \in \mathcal{H}$.

From (19) and (27) we have for $y = x$ with $\|x\| = 1$ that

$$\begin{aligned}
&\omega^2 \left(\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i \right) \\
&= \sup_{\|x\|=1} \left| \left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, x \right\rangle \right|^2 \\
&\leq \sup_{\|x\|=1} \left[\left\langle \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left| |X_i|^\alpha U_i \right|^{2p} x, x \right\rangle^{\frac{1}{p}} \left\langle \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} y, y \right\rangle^{\frac{1}{q}} \right] \\
&\leq \sup_{\|x\|=1} \left\langle \sum_{i=1}^{n} p_i \left(\frac{1}{p} h_a^p(\|X_i\|) \left| |X_i|^\alpha U_i \right|^{2p} + \frac{1}{q} h_a^q(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} \right) x, x \right\rangle \\
&= \left\| \sum_{i=1}^{n} p_i \left(\frac{1}{p} h_a^p(\|X_i\|) \left| |X_i|^\alpha U_i \right|^{2p} + \frac{1}{q} h_a^q(\|X_i\|) \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} \right) \right\|,
\end{aligned}$$

which proves (26).

□

In the following remark, we present a special case of Theorem 2 that is particularly interesting.

Remark 12. *If we take $h \equiv 1$ in Theorem 2, then we obtain*

$$\left\| \sum_{i=1}^{n} p_i V_i^* X_i U_i \right\|^2 \leq \left\| \sum_{i=1}^{n} p_i \left| |X_i|^\alpha U_i \right|^{2p} \right\|^{\frac{1}{p}} \left\| \sum_{i=1}^{n} p_i h \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} \right\|^{\frac{1}{q}}$$

and

$$\omega^2 \left(\sum_{i=1}^{n} p_i V_i^* X_i U_i \right) \leq \left\| \sum_{i=1}^{n} p_i \left(\frac{1}{p} \left| |X_i|^\alpha U_i \right|^{2p} + \frac{1}{q} \left| |X_i^*|^{1-\alpha} V_i \right|^{2q} \right) \right\|.$$

The case for two operators outlined in more details in the introduction, is as follows:

$$\|A + B\|^2 \leq \left\| |A|^{2\alpha p} + |B|^{2\alpha p} \right\|^{\frac{1}{p}} \left\| |A^*|^{2(1-\alpha)q} + |B^*|^{2(1-\alpha)q} \right\|^{\frac{1}{q}}$$

and

$$\omega^2(A + B) \leq \left\| \frac{1}{p} \left(|A|^{2\alpha p} + |B|^{2\alpha p} \right) + \frac{1}{q} \left(|A^*|^{2(1-\alpha)q} + |B^*|^{2(1-\alpha)q} \right) \right\|$$

.for $A, B \in \mathcal{L}(\mathbb{H})$, $\alpha \in [0,1]$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

As a direct consequence of Theorem 2, we obtain the following corollaries.

Corollary 2. *Let $h(z) := \sum_{k=0}^{\infty} a_k z^k$ be a convergent power series with complex coefficients on $D(0,\rho)$. Take X_i, U_i, $V_i \in \mathcal{L}(\mathbb{H})$ with $\|X_i\| < \rho$ for $i \in \{1,\ldots,n\}$. Choose $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for non-negative p_i ($i = 1$ to n) with $\sum_{i=1}^{n} p_i > 0$, the following inequalities hold*

$$\begin{aligned}\left\|\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right\|^2 &\le \left\|\sum_{i=1}^{n} p_i h_a^p(\|X_i\|)\left||X_i|^{\frac{1}{2}} U_i\right|^{2p}\right\|^{\frac{1}{p}} \left\|\sum_{i=1}^{n} p_i h_a^q(\|X_i\|)\left||X_i^*|^{\frac{1}{2}} V_i\right|^{2q}\right\|^{\frac{1}{q}} \\ &\le \max_{k=1,\ldots,n} h_a^2(\|X_k\|) \left\|\sum_{i=1}^{n} p_i \left||X_i|^{\frac{1}{2}} U_i\right|^{2p}\right\|^{\frac{1}{p}} \left\|\sum_{i=1}^{n} p_i \left||X_i^*|^{\frac{1}{2}} V_i\right|^{2q}\right\|^{\frac{1}{q}},\end{aligned}$$

and

$$\omega^2\left(\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right) \le \left\|\sum_{i=1}^{n} p_i \left(\frac{1}{p} h_a^p(\|X_i\|)\left||X_i|^{\frac{1}{2}} U_i\right|^{2p} + \frac{1}{q} h_a^q(\|X_i\|)\left||X_i^*|^{\frac{1}{2}} V_i\right|^{2q}\right)\right\|.$$

Corollary 3. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}\omega^2\left(\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right) &\le \frac{1}{2}\left\|\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)\left(\left||X_i|^{\alpha} U_i\right|^4 + \left||X_i^*|^{1-\alpha} V_i\right|^4\right)\right\| \\ &\le \frac{1}{2} \max_{k=1,\ldots,n} h_a^2(\|X_k\|)\left\|\sum_{i=1}^{n} p_i \left(\left||X_i|^{\alpha} U_i\right|^4 + \left||X_i^*|^{1-\alpha} V_i\right|^4\right)\right\|.\end{aligned}$$

In particular,

$$\begin{aligned}\omega^2\left(\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right) &\le \frac{1}{2}\left\|\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)\left(\left||X_i|^{\frac{1}{2}} U_i\right|^4 + \left||X_i^*|^{\frac{1}{2}} V_i\right|^4\right)\right\| \\ &\le \frac{1}{2} \max_{k=1,\ldots,n} h_a^2(\|X_k\|)\left\|\sum_{i=1}^{n} p_i \left(\left||X_i|^{\frac{1}{2}} U_i\right|^4 + \left||X_i^*|^{\frac{1}{2}} V_i\right|^4\right)\right\|.\end{aligned}$$

The following remark shows significant consequences and examples from previous findings.

Remark 13. *(1) If we take $V_i = U_i = I$, then for $p_i \ge 0$, $i \in \{1,\ldots,n\}$ with $\sum_{i=1}^{n} p_i > 0$, we obtain from Corollary 3*

$$\begin{aligned}\omega^2\left(\sum_{i=1}^{n} p_i X_i h(X_i)\right) &\le \frac{1}{2}\left\|\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)\left(|X_i|^{4\alpha} + |X_i^*|^{4(1-\alpha)}\right)\right\| \\ &\le \frac{1}{2} \max_{k=1,\ldots,n} h_a^2(\|X_k\|)\left\|\sum_{i=1}^{n} p_i \left(|X_i|^{4\alpha} + |X_i^*|^{4(1-\alpha)}\right)\right\|\end{aligned} \tag{28}$$

for all $\alpha \in [0,1]$. *In particular,*

$$\omega^2\left(\sum_{i=1}^n p_i X_i h(X_i)\right) \le \frac{1}{2}\left\|\sum_{i=1}^n p_i h_a^2(\|X_i\|)\left(|X_i|^2 + |X_i^*|^2\right)\right\| \le \frac{1}{2}\max_{k=1,\ldots,n} h_a^2(\|X_k\|)\left\|\sum_{i=1}^n p_i\left(|X_i|^2 + |X_i^*|^2\right)\right\|.$$

(2) *Now, if we take, for instance the resolvent function* $h(\mu) = (1\pm\mu)^{-1}$ *with* $|\mu| < 1$*, then we obtain from* (28) *that*

$$\omega^2\left(\sum_{i=1}^n p_i X_i (1\pm X_i)^{-1}\right) \le \frac{1}{2}\left\|\sum_{i=1}^n p_i (1-\|X_i\|)^{-2}\left(|X_i|^{4\alpha} + |X_i^*|^{4(1-\alpha)}\right)\right\| \le \frac{1}{2}\left(1-\max_{k=1,\ldots,n}\|X_k\|\right)^{-2}\left\|\sum_{i=1}^n p_i\left(|X_i|^{4\alpha} + |X_i^*|^{4(1-\alpha)}\right)\right\|$$

for $\|X_i\| < 1$, $i = 1, \ldots, n$.

We also have the following result concerning the powers of numerical radius:

Theorem 3. *With the assumptions of Theorem* 2 *and if* $r \ge 1$*, then*

$$\omega^{2r}\left(\sum_{i=1}^n p_i V_i^* X_i h(X_i) U_i\right) \le \left\|\frac{1}{p}\left(\sum_{i=1}^n p_i h_a^p(\|X_i\|)\left||X_i|^\alpha U_i\right|^{2p}\right)^r + \frac{1}{q}\left(\sum_{i=1}^n p_i h_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^r\right\|. \tag{29}$$

Also, if $s \ge \max\{p,q\} > 1$*, then*

$$\omega^{2s}\left(\sum_{i=1}^n p_i V_i^* X_i h(X_i) U_i\right) \le \frac{1}{2}\left\|\sum_{i=1}^n p_i h_a^p(\|X_i\|)\left||X_i|^\alpha U_i\right|^{2p}\right\|^{\frac{s}{p}}\left\|\sum_{i=1}^n p_i h_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right\|^{\frac{s}{q}} + \frac{1}{2}\omega\left[\left(\sum_{i=1}^n p_i h_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^{\frac{s}{q}}\left(\sum_{i=1}^n p_i h_a^p(\|X_i\|)\left||X_i|^\alpha U_i\right|^{2p}\right)^{\frac{s}{p}}\right]. \tag{30}$$

Proof. From (19) we obtain for $y = x$ with $\|x\| = 1$ that

$$\left|\left\langle \sum_{i=1}^n p_i V_i^* X_i h(X_i) U_i x, x\right\rangle\right|^2 \le \left\langle \sum_{i=1}^n p_i h_a^p(\|X_i\|)\left||X_i|^\alpha U_i\right|^{2p} x, x\right\rangle^{\frac{1}{p}}\left\langle \sum_{i=1}^n p_i h_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^{2q} x, x\right\rangle^{\frac{1}{q}}. \tag{31}$$

If we take the power $r \geq 1$ and use McCarthy's inequality, then we have

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, x \right\rangle\right|^{2r} \\
&\leq \left\langle \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^\alpha U_i\right|^{2p} x, x \right\rangle^{r/p} \left\langle \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q} x, x \right\rangle^{r/q} \\
&\leq \left\langle \left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^\alpha U_i\right|^{2p}\right)^r x, x \right\rangle^{\frac{1}{p}} \left\langle \left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^r x, x \right\rangle^{\frac{1}{q}}
\end{aligned} \tag{32}$$

for $x \in \mathcal{H}$, $\|x\| = 1$.

Using Young's inequality we also have

$$\begin{aligned}
&\left\langle \left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^\alpha U_i\right|^{2p}\right)^r x, x \right\rangle^{\frac{1}{p}} \left\langle \left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^r x, x \right\rangle^{\frac{1}{q}} \\
&\leq \frac{1}{p} \left\langle \left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^\alpha U_i\right|^{2p}\right)^r x, x \right\rangle + \frac{1}{q} \left\langle \left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^r x, x \right\rangle \\
&\quad = \left\langle \left[\frac{1}{p}\left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^\alpha U_i\right|^{2p}\right)^r + \frac{1}{q}\left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^r\right] x, x \right\rangle,
\end{aligned}$$

which, by (32) gives

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, x \right\rangle\right|^{2r} \\
&\qquad \leq \left\langle \left[\frac{1}{p}\left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^\alpha U_i\right|^{2p}\right)^r + \frac{1}{q}\left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^r\right] x, x \right\rangle
\end{aligned}$$

for $x \in \mathcal{H}$, $\|x\| = 1$.

If we take the supremum over $\|x\| = 1$, then we obtain the desired result (29).

From (31) and McCarthy's inequality we have

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i x, x \right\rangle\right|^{2s} \\
&\leq \left\langle \sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^\alpha U_i\right|^{2p} x, x \right\rangle^{\frac{s}{p}} \left\langle \sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q} x, x \right\rangle^{\frac{s}{q}} \\
&\leq \left\langle \left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|) \left||X_i|^\alpha U_i\right|^{2p}\right)^{\frac{s}{p}} x, x \right\rangle \left\langle \left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|) \left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^{\frac{s}{q}} x, x \right\rangle
\end{aligned} \tag{33}$$

for $x \in \mathcal{H}$, $\|x\| = 1$.

From Buzano's inequality, we also have

$$\begin{aligned}
&\left\langle\left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|)\left||X_i|^\alpha U_i\right|^{2p}\right)^{\frac{s}{p}} x, x\right\rangle\left\langle x, \left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^{\frac{s}{q}} x\right\rangle \\
&\leq \frac{1}{2}\left\|\left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|)\left||X_i|^\alpha U_i\right|^{2p}\right)^{\frac{s}{p}} x\right\|\left\|\left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^{\frac{s}{q}} x\right\| \\
&+\frac{1}{2}\left|\left\langle\left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|)\left||X_i|^\alpha U_i\right|^{2p}\right)^{\frac{s}{p}} x, \left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^{\frac{s}{q}} x\right\rangle\right| \\
&=\frac{1}{2}\left\|\left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|)\left||X_i|^\alpha U_i\right|^{2p}\right)^{\frac{s}{p}} x\right\|\left\|\left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^{\frac{s}{q}} x\right\| \\
&+\frac{1}{2}\left|\left\langle\left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^{\frac{s}{q}}\left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|)\left||X_i|^\alpha U_i\right|^{2p}\right)^{\frac{s}{p}} x, x\right\rangle\right| \qquad (34)
\end{aligned}$$

for $x \in \mathcal{H}$, $\|x\| = 1$.

By utilizing (33) and (34) and then taking the supremum over $\|x\| = 1$, we obtain (30). □

Theorem 3 provides us important insights and implications, leading to some interesting remarks and consequences. By carefully studying the theorem, we can discover the following remarks and corollary, which help us understand the topic even better.

Remark 14. *It is worth noting that an interesting consequence can be observed by considering the special case where $h \equiv 1$ in Theorem 3. By doing so, we obtain the following result:*

$$\begin{aligned}
&\omega^{2r}\left(\sum_{i=1}^{n} p_i V_i^* X_i U_i\right) \\
&\leq \left\|\frac{1}{p}\left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|)\left||X_i|^\alpha U_i\right|^{2p}\right)^{r} + \frac{1}{q}\left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^{r}\right\|.
\end{aligned}$$

and, if $s \geq \max\{p, q\} > 1$, then

$$\begin{aligned}
&\omega^{2s}\left(\sum_{i=1}^{n} p_i V_i^* X_i U_i\right) \\
&\leq \frac{1}{2}\left\|\sum_{i=1}^{n} p_i\left||X_i|^\alpha U_i\right|^{2p}\right\|^{\frac{s}{p}}\left\|\sum_{i=1}^{n} p_i\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right\|^{\frac{s}{q}} \\
&+\frac{1}{2}\omega\left[\left(\sum_{i=1}^{n} p_i\left||X_i^*|^{1-\alpha} V_i\right|^{2q}\right)^{\frac{s}{q}}\left(\sum_{i=1}^{n} p_i\left||X_i|^\alpha U_i\right|^{2p}\right)^{\frac{s}{p}}\right].
\end{aligned}$$

Remark 15. *By letting $\alpha = \frac{1}{2}$ in Theorem 3, we deduce that*

$$\begin{aligned}
&\omega^{2r}\left(\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right) \\
&\leq \left\|\frac{1}{p}\left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|)\left||X_i|^{\frac{1}{2}} U_i\right|^{2p}\right)^{r} + \frac{1}{q}\left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|)\left||X_i^*|^{\frac{1}{2}} V_i\right|^{2q}\right)^{r}\right\|,
\end{aligned}$$

and

$$\begin{aligned}
&\omega^{2s}\left(\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right) \\
&\qquad \leq \frac{1}{2}\left\|\sum_{i=1}^{n} p_i h_a^p(\|X_i\|)\left||X_i|^{\frac{1}{2}} U_i\right|^{2p}\right\|^{\frac{s}{p}}\left\|\sum_{i=1}^{n} p_i h_a^q(\|X_i\|)\left||X_i^*|^{\frac{1}{2}} V_i\right|^{2q}\right\|^{\frac{s}{q}} \\
&\qquad\quad + \frac{1}{2}\omega\left[\left(\sum_{i=1}^{n} p_i h_a^q(\|X_i\|)\left||X_i^*|^{\frac{1}{2}} V_i\right|^{2q}\right)^{\frac{s}{q}}\left(\sum_{i=1}^{n} p_i h_a^p(\|X_i\|)\left||X_i|^{\frac{1}{2}} U_i\right|^{2p}\right)^{\frac{s}{p}}\right].
\end{aligned}$$

Corollary 4. *With the assumptions of Theorem 2, we have for $r \geq 1$ that*

$$\begin{aligned}
&\omega^{2r}\left(\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right) \\
&\leq \frac{1}{2}\left\|\left(\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)\left||X_i|^{\alpha} U_i\right|^4\right)^r + \left(\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)\left||X_i^*|^{1-\alpha} V_i\right|^4\right)^r\right\| \\
&\leq \frac{1}{2}\max_{k=1,\ldots,n} h_a^{2r}(\|X_k\|)\left\|\left(\sum_{i=1}^{n} p_i\left||X_i|^{\alpha} U_i\right|^4\right)^r + \left(\sum_{i=1}^{n} p_i\left||X_i^*|^{1-\alpha} V_i\right|^4\right)^r\right\|.
\end{aligned}$$

In particular,

$$\begin{aligned}
&\omega^{2r}\left(\sum_{i=1}^{n} p_i V_i^* X_i h(X_i) U_i\right) \\
&\leq \frac{1}{2}\left\|\left(\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)\left||X_i|^{\frac{1}{2}} U_i\right|^4\right)^r + \left(\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)\left||X_i^*|^{\frac{1}{2}} V_i\right|^4\right)^r\right\| \\
&\leq \frac{1}{2}\max_{k=1,\ldots,n} h_a^{2r}(\|X_k\|)\left\|\left(\sum_{i=1}^{n} p_i\left||X_i|^{\frac{1}{2}} U_i\right|^4\right)^r + \left(\sum_{i=1}^{n} p_i\left||X_i^*|^{\frac{1}{2}} V_i\right|^4\right)^r\right\|.
\end{aligned}$$

Remark 16. *(1) If we take $V_i = U_i = I$ then for $p_i \geq 0$, $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^{n} p_i > 0$, we get from Corollary 4 that*

$$\begin{aligned}
&\omega^{2r}\left(\sum_{i=1}^{n} p_i X_i h(X_i)\right) \qquad (35) \\
&\leq \frac{1}{2}\left\|\left(\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)|X_i|^{4\alpha}\right)^r + \left(\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)|X_i^*|^{4(1-\alpha)}\right)^r\right\| \\
&\leq \frac{1}{2}\max_{k=1,\ldots,n} h_a^{2r}(\|X_k\|)\left\|\left(\sum_{i=1}^{n} p_i|X_i|^{4\alpha}\right)^r + \left(\sum_{i=1}^{n} p_i|X_i^*|^{4(1-\alpha)}\right)^r\right\|.
\end{aligned}$$

In particular,

$$\begin{aligned}
&\omega^{2r}\left(\sum_{i=1}^{n} p_i X_i h(X_i)\right) \\
&\leq \frac{1}{2}\left\|\left(\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)|X_i|^2\right)^r + \left(\sum_{i=1}^{n} p_i h_a^2(\|X_i\|)|X_i^*|^2\right)^r\right\| \\
&\leq \frac{1}{2}\max_{k=1,\ldots,n} h_a^{2r}(\|X_k\|)\left\|\left(\sum_{i=1}^{n} p_i|X_i|^2\right)^r + \left(\sum_{i=1}^{n} p_i|X_i^*|^2\right)^r\right\|.
\end{aligned}$$

(2) Now, if we take, for instance $h(\mu) = (1 \pm \mu)^{-1}$ *with* $|\mu| < 1$, *then we obtain from (35) that*

$$\begin{aligned}
&\omega^{2r}\left(\sum_{i=1}^{n} p_i X_i (1 \pm X_i)^{-1}\right) \\
&\leq \frac{1}{2}\left\| \left(\sum_{i=1}^{n} p_i (1 \pm \|X_i\|)^{-2} |X_i|^{4\alpha}\right)^r + \left(\sum_{i=1}^{n} p_i (1 \pm \|X_i\|)^{-2} |X_i^*|^{4(1-\alpha)}\right)^r \right\| \\
&\leq \frac{1}{2}\left(1 \pm \max_{k=1,\ldots,n} \|X_k\|\right)^{-2r} \left\| \left(\sum_{i=1}^{n} p_i |X_i|^{4\alpha}\right)^r + \left(\sum_{i=1}^{n} p_i |X_i^*|^{4(1-\alpha)}\right)^r \right\|
\end{aligned}$$

for $\|X_i\| < 1$, $i = 1, \ldots, n$.

Various similar results for other fundamental complex functions such as, the logarithm function, the complex exponential, the complex trigonometric, and hyperbolic functions can be stated as well. The details are omitted.

4. Conclusions

In summary, this paper explores power series in Hilbert spaces. We focused on series like $h(z) = \sum_{k=0}^{\infty} a_k z^k$ and its modified version $h_a(z) = \sum_{k=0}^{\infty} |a_k| z^k$, where a_k are complex numbers. By using Hölder-type inequalities, we found different inequalities for operators that work on these series. We made these discoveries assuming that $h(z)$ converges on the open disk $D(0, \rho)$, where ρ is the radius of convergence.

We also explored norm and numerical radius inequalities related to these power series. Our main goal in this paper was to improve our understanding of mathematical inequalities and help others learn more about them. Our work is an important step forward in theory, offering new ideas and tools for mathematicians in this field.

The inequalities we found can be useful for analyzing various properties of power series and how they are used in functional analysis and related areas. They provide a good starting point for more research and help us understand how power series behave in Hilbert spaces. By learning more about mathematical inequalities, we can help advance mathematics and find new applications for these ideas in the future.

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