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Bullen-Mercer type inequalities with applications in numerical analysis

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ABSTRACT

In mathematical analysis theory of inequalities has considerable influence due to its massive utility in various fields of physical sciences. These are investigated via multiple approaches to acquire more precise and rectified forms of already celebrated consequences. Integral inequalities are investigated to compute the error bounds for quadrature schemes. Among all of them, one is Hermite-Hadamard inequality, which has mighty efficacy. Numerous generalizations have been proposed in the literature based on different novel and innovative procedures. In recent years, Bullen inequality has been very commonly studied inequality. The main objective of our progressive study is to establish a new set of Bullen-type inequalities concerning the Jensen-Mecer inequality. For the completion of the current investigation, we derive a new general Bullen-Mecer equality, which is beneficial to achieve our primary consequences. Furthermore, Considering the Bullen-Mecer equation, we employ the convexity property together with famous Hölder's type and Young's inequalities, bounding, and Lipschitz characteristics of functions to conclude new variants of generalized upper bounds of Bullen inequality. Also, we deliver some applications of outcomes to means, special functions, error bounds, and iterative methods to solve non-linear problems. Lastly, we verify our findings through various simulations. The advantage of the current study is that several results concerning Bullen's inequality can be retrieved from our proposed results and various new results can be achieved by choosing the values for γ and δ. By utilizing the similar technique that we have adopted new iterative schemes can be established from integral inequalities.

1. Introduction

A set $C \subset \mathbb{R}$ is said to be a convex if

$$(1-\omega)\rho_1 + \omega\rho_2 \in C, \quad \forall \rho_1, \rho_2 \in C, \omega \in [0,1].$$

Likewise, a function $\Psi : C \to \mathbb{R}$ is considered convex if

$$\Psi((1-\omega)\rho_1 + \omega\rho_2) \leq (1-\omega)\Psi(\rho_1) + \omega\Psi(\rho_2), \quad \forall \rho_1, \rho_2 \in C, \omega \in [0,1].$$

For more details, see [1].

The notion of convexity has been extensively applied to establish integral inequalities. We revisit a well-known inequality, which is described in [2] as:

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If $\Psi : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function, then

$$\Psi\left(\frac{\rho_1+\rho_2}{2}\right) \leq \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} \Psi(\delta)\mathrm{d}\delta \leq \frac{\Psi(\rho_1)+\Psi(\rho_2)}{2}.$$

Now we present another famous result due to Jensen, which generalizes the idea of convex mappings and is described in [2] as: Suppose Ψ is a convex mapping defined on $[\rho_1, \rho_2]$, then for any $\delta_i \in [\rho_1, \rho_2]$ and $\mu_i \in [0,1]$, with $i = 1, 2, \dots, n$ and $\sum_{i=1}^{n} \mu_i = 1$, the following inequality holds:

$$\Psi\left(\sum_{i=1}^{n} \mu_i \delta_i\right) \leq \sum_{i=1}^{n} \mu_i \Psi(\delta_i).$$

For more details, see [3].

In 2004, Mercer [4] established an improved version of Jensen's inequality which is described as follows:

If $\Psi : [\rho_1, \rho_2] \to \mathbb{R}$ is a convex function and $\rho_1 < \delta < \gamma < \rho_2$, then the following inequality holds:

$$\Psi\left(\rho_1+\rho_2-\sum_{i=1}^{n} \mu_i \delta_i\right) \leq \Psi(\rho_1)+\Psi(\rho_2)-\sum_{i=1}^{n} \mu_i \Psi(\delta_i).$$

Integral inequalities are considered from numerous perspectives, but one of its areas is linked with error estimations of commonly studied quadrature and cubature rules.

In [5] Bullen explored another Hermite-Hadamard type inequality, which is regarded as Bullen's inequality, Suppose $\Psi : [\rho_1, \rho_2] \to \mathbb{R}$ be a convex mapping then

$$\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} \Psi(u)\mathrm{d}u \leq \frac{1}{2}\left[\Psi\left(\frac{\rho_1+\rho_2}{2}\right)+\frac{\Psi(\rho_1)+\Psi(\rho_2)}{2}\right].$$

In recent many notable developments regarding this inequality have been established in the literature. This inequality provides the error bounds for the remainder of Bullen quadrature schemes. The main purpose of investigating such types of inequalities is to predict more accurate and strong bounds of error terms. In [6,7] Cakmak derived different fractional error estimates via various classes of convexity and also examined the Bullen-like variants involving conformable fractional operators and convex mappings. Erden and Sarikaya [8] explored the Bullen-like inequalities within the fractal domain. In [9] authors have successfully utilized the unified fractional operators to construct new bounds for Bullen results. Zhao et al. [10] visualized the fractional versions of Bullen result based on another general identity. Hezenci and his colleagues [11] studied the Bullen-type inequalities involving conformable integral operators. In [12] Boulares and his fellows obtained new fractional multiplicative counterparts of Bullen inequality in associated with convex functions. In [13] Agarwal and Tomar utilized the general family of fractional operators approach to extend Hadamard's type integral inequalities. In [14] Agarwal explored the trapezium-like inequalities in the setting of k-fractional operators. Ali et al. [15] explored the new quantum estimates for Simpson's and Newton's quadrature schemes through preinvex mappings. For more detail see [16–19].

We recollect some Hölders type inequalities, which will play a crucial role in the development of better bounds.

Theorem 1.1. *(Hölder's Inequality [20]). Let $\Psi, \Phi : [\rho_1, \rho_2] \to \mathbb{d}u$ and $|\Psi|^p, |\Phi|^q \in L[\rho_1, \rho_2]$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\int_{\rho_1}^{\rho_2} |\Psi(u)\Phi(u)|\mathrm{d}u \leq \left(\int_{\rho_1}^{\rho_2} |\Psi(u)|^p\right)^{\frac{1}{p}} \left(\int_{\rho_1}^{\rho_2} |\Phi(u)|^q \mathrm{d}u\right)^{\frac{1}{q}}.$$

Theorem 1.2. *(Improved Hölder's Inequality [20]). Let $\Psi, \Phi : [\rho_1, \rho_2] \to \mathbb{d}u$ and $|\Psi|^p, |\Phi|^q \in L[\rho_1, \rho_2]$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}\int_{\rho_1}^{\rho_2} |\Psi(u)\Phi(u)|\mathrm{d}u &\leq \frac{1}{\rho_2-\rho_1}\left(\int_{\rho_1}^{\rho_2} |(\rho_2-u)\Psi(u)|^p\right)^{\frac{1}{p}} \left(\int_{\rho_1}^{\rho_2} (\rho_2-u)|\Phi(u)|^q \mathrm{d}u\right)^{\frac{1}{q}} \\ &+ \frac{1}{\rho_2-\rho_1}\left(\int_{\rho_1}^{\rho_2} |(u-\rho_1)\Psi(u)|^p\right)^{\frac{1}{p}} \left(\int_{\rho_1}^{\rho_2} (u-\rho_1)|\Phi(u)|^q \mathrm{d}u\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 1.3. *(Power mean's Inequality [20]). Let $\Psi, \Phi : [\rho_1, \rho_2] \to \mathbb{d}u$ and $|\Psi|^p, |\Phi|^q \in L[\rho_1, \rho_2]$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\int_{\rho_1}^{\rho_2} |\Psi(u)\Phi(u)|\mathrm{d}u \leq \left(\int_{\rho_1}^{\rho_2} |\Psi(u)|\right)^{1-\frac{1}{q}} \left(\int_{\rho_1}^{\rho_2} |\Psi(u)||\Phi(u)|^q \mathrm{d}u\right)^{\frac{1}{q}}.$$

Theorem 1.4. *(Improved power mean's Inequality [21]). Let $\Psi, \Phi : [\rho_1, \rho_2] \to \mathbb{d}u$ and $|\Psi|^p, |\Phi|^q \in L[\rho_1, \rho_2]$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_{\rho_1}^{\rho_2} |\Psi(u)\Phi(u)|\mathrm{d}u \leq \frac{1}{\rho_2 - \rho_1}\left(\int_{\rho_1}^{\rho_2} |(\rho_2 - u)\Psi(u)|\right)^{1-\frac{1}{q}} \left(\int_{\rho_1}^{\rho_2} (\rho_2 - u)|\Psi(u)||\Phi(u)|^q \mathrm{d}u\right)^{\frac{1}{q}}$$
$$+ \frac{1}{\rho_2 - \rho_1}\left(\int_{\rho_1}^{\rho_2} |(u - \rho_1)\Psi(u)|\right)^{\frac{1}{p}} \left(\int_{\rho_1}^{\rho_2} (u - \rho_1)|\Psi(u)||\Phi(u)|^q \mathrm{d}u\right)^{\frac{1}{q}}.$$

First time, Ogulmus et al. [22] examined trapezoidal type inequalities involving Mercer-type inequality and the classical Riemann-Liouville fractional operators. Iscan et al. [23] explored the idea of [22] by making weight functions. Finally, You et al. [24] studied the Hermite-Hadamard-Mercer type inequalities invoking the idea of harmonic convex functions.

Cortez et al. [25] computed the Hermite-Hadamard-Mercer type inequalities incorporated with the general version of Mercer inequality for convex functions and k fractional variants along with their applications. Butt et al. [26] implemented the Mercer inequality and unified integral operator concerning a monotone function to establish some counterparts of Ostrowski's Inequality. Recently, in 2022, Faisal et al. [27] investigated both the continuous and discrete form of Hermite-Hadamard-Mercer type inequalities in the setting of majorization theory and Mercer inequality proposed by Neizgoda. In [28], Bin-Mohsin et al. analyzed some strong bounds of trapezoidal-Mercer inequality from the perspective of fractional calculus. In 2022, Liu et al. [29] concluded some fractional variants of Mercer-type inequalities implementing the notion of convex mappings and AB-integral operators. Moreover, Budak et al. [30] computed the error bounds of Milne inequality in a fractional form involving convex mappings and conducted numeric verifications and simulations. Recently, Meftah et al. [31] analyzed the error estimation of the Milne formula in the fractal domain via generalized convexity. Ali et al. [32] explored new fractional equivalents of Milnes-formula through convex mappings. Also, in [33] authors investigated the Milne-Mercer type schemes in the context of quantum calculus. For more details, see [34], [35], and [36].

The main motivation of this paper is that we will establish some new generalizations of classical Bullen's type inequalities involving the idea of Jensen-Mercer inequality for convex functions and some novel applications to numerical analysis and special functions. To achieve our desired outcomes, we have divided our study into four sections. In the first section, we recollect some essential notions and facts that are fruitful for further investigation. The second section consists of primary findings. To prove our main results, first, we establish a new equation named Bullen-Mercer identity and by taking into account the Bullen-Mercer identity together with a discrete form of Jensen-Mercer inequality aided with several well-known inequalities, we will acquire new estimates of Bullen-type inequalities. In the next part, we present some interesting and novel applications to special means, error bounds, special functions, and especially iterative methods to solve non-linear problems. Finally, in the last section, we will present some graphical analysis to verify the correctness of our primary outcomes.

2. Main results

Lemma 2.1. *Suppose that $\Psi : [\rho_1, \rho_2] \to \mathbb{R}$ be differentiable functions over (ρ_1, ρ_2) with $\rho_1 < \delta < \gamma < \rho_2$. Then,*

$$Z(\rho_1, \rho_2, \delta, \gamma) = \frac{(\gamma - \delta)}{8}\left[\int_0^1 (1 - 2\omega)\Psi'\left(\rho_1 + \rho_2 - \frac{1-\omega}{2}\delta - \frac{1+\omega}{2}\gamma\right)\mathrm{d}\omega - \int_0^1 (1 - 2\omega)\Psi'\left(\rho_1 + \rho_2 - \frac{1+\omega}{2}\delta - \frac{1-\omega}{2}\gamma\right)\mathrm{d}\omega\right], \tag{2.1}$$

where

$$Z(\rho_1, \rho_2, \delta, \gamma) = \frac{1}{2}\left[\frac{\Psi(\rho_1 + \rho_2 - \delta) + \Psi(\rho_1 + \rho_2 - \gamma)}{2} + \Psi\left(\rho_1 + \rho_2 - \frac{\delta+\gamma}{2}\right)\right] - \frac{1}{\gamma - \delta}\int_{\rho_1+\rho_2-\gamma}^{\rho_1+\rho_2-\delta} \Psi(u)\mathrm{d}u.$$

Proof. Considering the right hand side of (2.1),

$$I = \frac{(\gamma - \delta)}{8}\left[\int_0^1 (1 - 2\omega)\Psi'\left(\rho_1 + \rho_2 - \frac{1-\omega}{2}\delta - \frac{1+\omega}{2}\gamma\right)\mathrm{d}\omega + \int_0^1 (2\omega - 1)\Psi'\left(\rho_1 + \rho_2 - \frac{1+\omega}{2}\delta - \frac{1-\omega}{2}\gamma\right)\mathrm{d}\omega\right]$$
$$= \frac{(\gamma - \delta)}{8}\left[I_1 + I_2\right], \tag{2.2}$$

where

$$I_1 = \int_0^1 (1 - 2\omega)\Psi'\left(\rho_1 + \rho_2 - \frac{1-\omega}{2}\delta - \frac{1+\omega}{2}\gamma\right)\mathrm{d}\omega.$$

Now implementing the integration by parts in the above relation, we obtain

$$I_1 = \frac{-2(1-2\omega)}{\gamma-\delta}\Psi\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\Big|_0^1 - \frac{4}{\gamma-\delta}\int_0^1 \Psi\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\mathrm{d}\omega$$

$$= \frac{2}{(\gamma-\delta)}\left[\Psi(\rho_1+\rho_2-\gamma)+\Psi\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\right] - \frac{8}{(\gamma-\delta)^2}\int_{\rho_1+\rho_2-\gamma}^{\rho_1+\rho_2-\frac{\delta+\gamma}{2}} \Psi(u)\mathrm{d}u.$$

Similarly,

$$I_2 = \int_0^1 (2\omega-1)\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\mathrm{d}\omega$$

$$= \frac{2}{(\gamma-\delta)}\left[\Psi(\rho_1+\rho_2-\delta)+\Psi\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\right] - \frac{8}{(\gamma-\delta)^2}\int_{\rho_1+\rho_2-\frac{\delta+\gamma}{2}}^{\rho_1+\rho_2-\delta} \Psi(u)\mathrm{d}u.$$

Inserting the values of I_1 and I_2, in (2.2), we attain the required identity. □

3. Bounds for Bullen-Mercer type inequalities

Theorem 3.1. *Under the assumptions of Lemma 2.1. If $|\Psi'|$ is a convex function, then*

$$|Z(\rho_1,\rho_2,\delta,\gamma)| \le \frac{(\gamma-\delta)}{8}\left[|\Psi'(\rho_1)|+|\Psi'(\rho_2)| - \frac{|\Psi'(\delta)|+|\Psi'(\gamma)|}{2}\right].$$

Proof. Implementing the modulus property to Lemma 2.1 and then utilizing the convexity characteristic of the function $|\Psi'|$,

$$\begin{aligned}
&|Z(\rho_1,\rho_2,\delta,\gamma)| \\
&\le \frac{(\gamma-\delta)}{8}\left[\int_0^1 |1-2\omega|\left\{\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|+\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|\right\}\mathrm{d}\omega\right] \\
&\le \frac{(\gamma-\delta)}{8}\left[\int_0^{\frac12} (1-2\omega)\left\{\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|+\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|\right\}\mathrm{d}\omega\right. \\
&\left.+\int_{\frac12}^1 (2\omega-1)\left\{\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|+\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|\right\}\mathrm{d}\omega\right] \\
&\le \frac{(\gamma-\delta)}{8}\left[\int_0^{\frac12} (1-2\omega)\left\{2(|\Psi'(\rho_1)+|\Psi'(\rho_2)|)-|\Psi'(\delta)-|\Psi'(\gamma)|\right\}\mathrm{d}\omega\right. \\
&\left.+\int_{\frac12}^1 (2\omega-1)\left\{2(|\Psi'(\rho_1)+|\Psi'(\rho_2)|)-|\Psi'(\delta)-|\Psi'(\gamma)|\right\}\mathrm{d}\omega\right] \\
&\le \frac{(\gamma-\delta)}{8}\left[\int_0^{\frac12} 2(1-2\omega)\left\{2(|\Psi'(\rho_1)+|\Psi'(\rho_2)|)-|\Psi'(\delta)-|\Psi'(\gamma)|\right\}\mathrm{d}\omega\right. \\
&\left.+\int_0^1 (2\omega-1)\left\{2(|\Psi'(\rho_1)+|\Psi'(\rho_2)|)-|\Psi'(\delta)-|\Psi'(\gamma)|\right\}\mathrm{d}\omega\right] \\
&= \frac{(\gamma-\delta)}{8}\left[|\Psi'(\rho_1)|+|\Psi'(\rho_2)|-\frac{|\Psi'(\delta)|+|\Psi'(\gamma)|}{2}\right].
\end{aligned}$$

Hence the proof is completed. □

Corollary 3.1. *By specifying* $\gamma = \frac{\rho_1+2\rho_2}{3}$ *and* $\delta = \frac{2\rho_1+\rho_2}{3}$ *in Theorem 3.1, then we have*

$$\frac{1}{2}\left[\frac{\Psi\left(\frac{2\rho_1+\rho_2}{3}\right)+\Psi\left(\frac{\rho_1+2\rho_2}{3}\right)}{2}+\Psi\left(\frac{\rho_1+\rho_2}{2}\right)\right]-\frac{3}{\rho_2-\rho_1}\int\limits_{\frac{2\rho_1+\rho_2}{3}}^{\frac{\rho_1+2\rho_2}{3}}\Psi(u)\mathrm{d}u$$
$$=\frac{\rho_2-\rho_1}{48}[|\Psi'(\rho_1)|+|\Psi'(\rho_2)|].$$

Remark 3.1. By specifying $\gamma=\rho_2$ and $\delta=\rho_1$ in Theorem 3.1, then we obtain the following result obtained in [37].

$$\frac{1}{2}\left[\frac{\Psi(\rho_1)+\Psi(\rho_2)}{2}+\Psi\left(\frac{\rho_1+\rho_2}{2}\right)\right]-\frac{1}{\rho_2-\rho_1}\int\limits_{\rho_1}^{\rho_2}\Psi(u)\mathrm{d}u$$
$$=\frac{\rho_2-\rho_1}{16}[|\Psi'(\rho_1)|+|\Psi'(\rho_2)|].$$

Remark 3.2. We can obtain several new and novel integral inequalities by specifying various other values for γ and δ.

Theorem 3.2. *Assume that all the conditions of Lemma 2.1 hold. If* $|\Psi'|^q$ *is convex function, then*

$$|Z(\rho_1,\rho_2,\delta,\gamma)|$$
$$\le\frac{(\gamma-\delta)}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1}{4}|\Psi'(\delta)|^q-\frac{3}{4}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{3}{4}|\Psi'(\delta)|^q-\frac{1}{4}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Implementing the modulus property and Hölder's inequality to Lemma 2.1 and then utilizing the convexity characteristic of the function $|\Psi'|^q$, we have

$$|Z(\rho_1,\rho_2,\delta,\gamma)|$$
$$\le\frac{(\gamma-\delta)}{8}\left[\int\limits_0^1|1-2\omega|\left\{\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|+\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|\right\}\mathrm{d}\omega\right]$$
$$\le\frac{(\gamma-\delta)}{8}\left(\int\limits_0^1|1-2\omega|^p\mathrm{d}\omega\right)^{\frac{1}{p}}\left[\left(\int\limits_0^1\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|^q\mathrm{d}\omega\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int\limits_0^1\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|^q\mathrm{d}\omega\right)^{\frac{1}{q}}\right]$$
$$\le\frac{(\gamma-\delta)}{8}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left[\left(\int\limits_0^1\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1-\omega}{2}|\Psi'(\delta)|^q-\frac{1+\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int\limits_0^1\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1+\omega}{2}|\Psi'(\delta)|^q-\frac{1-\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}\right]$$
$$=\frac{(\gamma-\delta)}{8}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left[\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1}{4}|\Psi'(\delta)|^q-\frac{3}{4}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{3}{4}|\Psi'(\delta)|^q-\frac{1}{4}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\right],$$

where

$$\int_0^1 |1-2\omega|^p \mathrm{d}\omega = \int_0^{\frac{1}{2}} (1-2\omega)^p \mathrm{d}\omega + \int_{\frac{1}{2}}^1 (2\omega-1)^p \mathrm{d}\omega = \frac{1}{1+p}.$$

Hence, we acquire our required result. □

Theorem 3.3. *Assume that all the conditions of Lemma 2.1 hold. If $|\Psi'|^q$ is convex function, then*

$$\begin{aligned} &|Z(\rho_1,\rho_2,\delta,\gamma)| \\ &\le \frac{(\gamma-\delta)}{8}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{4}-\frac{1}{8}|\Psi'(\delta)|^q-\frac{3}{8}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{4}-\frac{3}{8}|\Psi'(\delta)|^q-\frac{1}{8}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\right], \end{aligned}$$

where $q \ge 1$.

Proof. Implementing the modulus property and Power mean's inequality on Lemma 2.1 and then utilizing the convexity characteristic of the function $|\Psi'|^q$, we have

$$\begin{aligned} &|Z(\rho_1,\rho_2,\delta,\gamma)| \\ &\le \frac{(\gamma-\delta)}{8}\left[\int_0^1 |1-2\omega|\left\{\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|+\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|\right\}\mathrm{d}\omega\right] \\ &\le \frac{(\gamma-\delta)}{8}\left(\int_0^1 |1-2\omega|\mathrm{d}\omega\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|^q\mathrm{d}\omega\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|^q\mathrm{d}\omega\right)^{\frac{1}{q}}\right] \\ &\le \frac{(\gamma-\delta)}{8}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 |1-2\omega|\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1-\omega}{2}|\Psi'(\delta)|^q-\frac{1+\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_0^1 |1-2\omega|\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1+\omega}{2}|\Psi'(\delta)|^q-\frac{1-\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}\right] \\ &= \frac{(\gamma-\delta)}{8}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{4}-\frac{1}{8}|\Psi'(\delta)|^q-\frac{3}{8}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{4}-\frac{3}{8}|\Psi'(\delta)|^q-\frac{1}{8}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Which is the required result. □

Theorem 3.4. *Assume that all the conditions of Lemma 2.1 hold. If $|\Psi'|^q$ and $|\Psi'|^p$ are convex functions, then*

$$|Z(\rho_1,\rho_2,\delta,\gamma)| \le \frac{(\gamma-\delta)}{8}\left[\frac{2}{p(p+1)}+\frac{2\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q\right)}{q}-\frac{|\Psi'(\delta)|^q+|\Psi'(\gamma)|^q}{q}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Implementing the modulus property and Young's inequality on Lemma 2.1 and then utilizing the convexity characteristic of the functions $|\Psi'|^p$ and $|\Psi'|^q$, we have

$$|Z(\rho_1,\rho_2,\delta,\gamma)|$$

$$\leq \frac{(\gamma-\delta)}{8}\left[\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|\mathrm{d}\omega\right.$$
$$\left.+\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|\mathrm{d}\omega\right]$$
$$\leq \frac{(\gamma-\delta)}{8}\left[\int_0^1\left(\frac{|1-2\omega|^p}{p}+\frac{\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|^q}{q}\right)\mathrm{d}\omega\right.$$
$$\left.+\int_0^1\left(\frac{|1-2\omega|^p}{p}+\frac{\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|^q}{q}\right)\mathrm{d}\omega\right]$$
$$\leq \frac{(\gamma-\delta)}{8}\left[2\int_0^1\frac{|1-2\omega|^p}{p}\mathrm{d}\omega+\int_0^1\left(\frac{\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|^q}{q}\right.\right.$$
$$\left.\left.+\frac{\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|^q}{q}\right)\mathrm{d}\omega\right]$$
$$\leq \frac{(\gamma-\delta)}{8}\left[\frac{2}{p(1+p)}+\int_0^1\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1-\omega}{2}|\Psi'(\delta)|^q-\frac{1+\omega}{2}|\Psi'(\gamma)|^q}{q}\right.\right.$$
$$\left.\left.+\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1+\omega}{2}|\Psi'(\delta)|^q-\frac{1-\omega}{2}|\Psi'(\gamma)|^q}{q}\right)\mathrm{d}\omega\right]$$
$$=\frac{(\gamma-\delta)}{8}\left[\frac{2}{p(p+1)}+\frac{2\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q\right)}{q}-\frac{|\Psi'(\delta)|^q+|\Psi'(\gamma)|^q}{q}\right].$$

This completes the proof. □

Theorem 3.5. *Assume that all the conditions of Lemma 2.1 hold. If $|\Psi'|^q$ is convex function, then*

$$|Z(\rho_1,\rho_2,\delta,\gamma)|$$
$$\leq \frac{(\gamma-\delta)}{8}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{2}-\frac{1}{6}|\Psi'(\delta)|^q-\frac{1}{3}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{2}-\frac{1}{3}|\Psi'(\delta)|^q-\frac{1}{6}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{2}-\frac{1}{12}|\Psi'(\delta)|^q-\frac{5}{12}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{2}-\frac{5}{12}|\Psi'(\delta)|^q-\frac{1}{12}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Implementing the modulus property and Holder-Iscan's inequality to Lemma 2.1 and then utilizing the convexity characteristic of the function $|\Psi'|^q$, we have

$$|Z(\rho_1,\rho_2,\delta,\gamma)|$$
$$\leq \frac{(\gamma-\delta)}{8}\left[\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|\mathrm{d}\omega\right.$$
$$\left.+\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|\mathrm{d}\omega\right]$$

$$\leq \frac{(\gamma-\delta)}{8}\Bigg[\left(\int_0^1 (1-\omega)|1-2\omega|^p \mathrm{d}\omega\right)^{\frac{1}{p}}\left(\int_0^1 (1-\omega)\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|^q \mathrm{d}\omega\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 (1-\omega)|1-2\omega|^p \mathrm{d}\omega\right)^{\frac{1}{p}}\left(\int_0^1 (1-\omega)\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|^q \mathrm{d}\omega\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \omega|1-2\omega|^p \mathrm{d}\omega\right)^{\frac{1}{p}}\left(\int_0^1 \omega\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|^q \mathrm{d}\omega\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \omega|1-2\omega|^p \mathrm{d}\omega\right)^{\frac{1}{p}}\left(\int_0^1 \omega\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|^q \mathrm{d}\omega\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{(\gamma-\delta)}{8}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\Bigg[\left(\int_0^1 (1-\omega)\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1-\omega}{2}|\Psi'(\delta)|^q-\frac{1+\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 (1-\omega)\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1+\omega}{2}|\Psi'(\delta)|^q-\frac{1-\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \omega\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1-\omega}{2}|\Psi'(\delta)|^q-\frac{1+\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \omega\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1+\omega}{2}|\Psi'(\delta)|^q-\frac{1-\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}\Bigg].$$

After simple computations, we acquire our result. □

Theorem 3.6. *Assume that all the conditions of Lemma 2.1 hold. If $|\Psi'|^q$ is convex function, then*

$$|Z(\rho_1,\rho_2,\delta,\gamma)|$$

$$\leq \frac{(\gamma-\delta)}{8}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\Bigg[\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{4}-\frac{3}{32}|\Psi'(\delta)|^q-\frac{5}{32}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{4}-\frac{5}{32}|\Psi'(\delta)|^q-\frac{3}{32}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{4}-\frac{1}{32}|\Psi'(\delta)|^q-\frac{7}{32}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q}{4}-\frac{7}{32}|\Psi'(\delta)|^q-\frac{1}{32}|\Psi'(\gamma)|^q\right)^{\frac{1}{q}}\Bigg],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Implementing the modulus property and improved Power-mean's inequality to Lemma 2.1 and then utilizing the convexity characteristic of the function $|\Psi'|^q$, we have

$$|Z(\rho_1,\rho_2,\delta,\gamma)|$$

$$\leq \frac{(\gamma-\delta)}{8}\Bigg[\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|\mathrm{d}\omega$$

$$+\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|\mathrm{d}\omega\Bigg]$$

$$\leq \frac{(\gamma-\delta)}{8}\left[\left(\int_0^1 (1-\omega)|1-2\omega|^p \mathrm{d}\omega\right)^{\frac{1}{p}}\left(\int_0^1 (1-\omega)|1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|^q \mathrm{d}\omega\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 (1-\omega)|1-2\omega| \mathrm{d}\omega\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\omega)|1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|^q \mathrm{d}\omega\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \omega|1-2\omega| \mathrm{d}\omega\right)^{1-\frac{1}{q}}\left(\int_0^1 \omega|1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|^q \mathrm{d}\omega\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1 \omega|1-2\omega| \mathrm{d}\omega\right)^{1-\frac{1}{q}}\left(\int_0^1 \omega|1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|^q \mathrm{d}\omega\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\gamma-\delta)}{8}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 (1-\omega)|1-2\omega|\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1-\omega}{2}|\Psi'(\delta)|^q-\frac{1+\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 (1-\omega)|1-2\omega|\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1+\omega}{2}|\Psi'(\delta)|^q-\frac{1-\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \omega|1-2\omega|\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1-\omega}{2}|\Psi'(\delta)|^q-\frac{1+\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1 \omega|1-2\omega|\left(|\Psi'(\rho_1)|^q+|\Psi'(\rho_2)|^q-\frac{1+\omega}{2}|\Psi'(\delta)|^q-\frac{1-\omega}{2}|\Psi'(\gamma)|^q\right)\mathrm{d}\omega\right)^{\frac{1}{q}}\right].$$

Some direct computations yield the required result. □

Theorem 3.7. *Assume that $\Psi : [\rho_1, \rho_2] \to \mathbb{R}$ be a Lipschitzian mapping with $L > 0$, then the following inequalities hold:*

$$\left|\frac{\Psi(\rho_1+\rho_2-\delta)+\Psi(\rho_1+\rho_2-\gamma)}{2}-\Psi\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\right| \leq \frac{L|\gamma-\delta|}{2}.$$

$$\left|\frac{1}{\gamma-\delta}\int_{\rho_1+\rho_2-\gamma}^{\rho_1+\rho_2-\delta}\Psi(u)\mathrm{d}u-\Psi\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\right| \leq \frac{L|\gamma-\delta|}{4}.$$

And

$$\left|\frac{\Psi(\rho_1+\rho_2-\delta)+\Psi(\rho_1+\rho_2-\gamma)}{2}-\frac{1}{\gamma-\delta}\int_{\rho_1+\rho_2-\gamma}^{\rho_1+\rho_2-\delta}\Psi(u)\mathrm{d}u\right| \leq \frac{L|\gamma-\delta|}{3}.$$

Proof. For any $\omega \in [0, 1]$, then

$$\begin{aligned}
&|\omega\Psi(\rho_1+\rho_2-\gamma)+(1-\omega)\Psi(\rho_1+\rho_2-\delta)-\Psi(\omega(\rho_1+\rho_2-\gamma)+(1-\omega)(\rho_1+\rho_2-\delta))|\\
&=\Big|\omega\Psi(\rho_1+\rho_2-\gamma)+(1-\omega)\Psi(\rho_1+\rho_2-\delta)-\big(\omega\Psi(\omega(\rho_1+\rho_2-\gamma)+(1-\omega)(\rho_1+\rho_2-\delta))\\
&\quad+(1-\omega)\Psi(\omega(\rho_1+\rho_2-\gamma)+(1-\omega)(\rho_1+\rho_2-\delta))\big)\Big|\\
&\leq \omega\left|\Psi(\rho_1+\rho_2-\gamma)-\Psi(\omega(\rho_1+\rho_2-\gamma)+(1-\omega)(\rho_1+\rho_2-\delta))\right|\\
&\quad+(1-\omega)\left|\Psi(\rho_1+\rho_2-\delta)-\Psi(\omega(\rho_1+\rho_2-\gamma)+(1-\omega)(\rho_1+\rho_2-\delta))\right|.
\end{aligned} \tag{3.1}$$

Now applying Lipschitzian condition on (3.1), we have

$$\begin{aligned}
&\omega\left|\Psi(\rho_1+\rho_2-\gamma)-\Psi(\omega(\rho_1+\rho_2-\gamma)+(1-\omega)(\rho_1+\rho_2-\delta))\right|\\
&+(1-\omega)\left|\Psi(\rho_1+\rho_2-\delta)-\Psi(\omega(\rho_1+\rho_2-\gamma)+(1-\omega)(\rho_1+\rho_2-\delta))\right|\\
&\leq \omega L\left|(1-\omega)(\gamma-\delta)\right|+(1-\omega)L\left|\omega(\gamma-\delta)\right|\\
&=2L\omega(1-\omega)|\gamma-\delta|
\end{aligned} \tag{3.2}$$

For $\omega = \frac{1}{2}$ in (3.2), we have

$$\left|\frac{\Psi(\rho_1+\rho_2-\delta)+\Psi(\rho_1+\rho_2-\gamma)}{2}-\Psi\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\right|\le\frac{L|\gamma-\delta|}{2}. \tag{3.3}$$

Substituting $\gamma=(1-\omega)\gamma+\omega\delta$ and $\delta=\omega\gamma+(1-\omega)\delta$ in (3.3), then

$$\left|\frac{\Psi\left(\rho_1+\rho_2-\omega\gamma-(1-\omega)\delta\right)+\Psi\left(\rho_1+\rho_2-(1-\omega)\gamma-\omega\delta\right)}{2}-\Psi\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\right|$$
$$\le\frac{L\,|1-2\omega|\,|\gamma-\delta|}{2}. \tag{3.4}$$

Now integrating the (3.4) with respect to $'\omega'$ over $[0,1]$, then

$$\left|\frac{1}{\gamma-\delta}\int_{\rho_1+\rho_2-\gamma}^{\rho_1+\rho_2-\delta}\Psi(u)\mathrm{d}u-\Psi\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\right|\le\frac{L|\gamma-\delta|}{4}.$$

Again from (3.2), we have

$$\left|\omega\Psi(\rho_1+\rho_2-\gamma)+(1-\omega)\Psi(\rho_1+\rho_2-\delta)-\Psi(\omega(\rho_1+\rho_2-\gamma)+(1-\omega)(\rho_1+\rho_2-\delta))\right|\le 2L\omega(1-\omega)|\gamma-\delta|. \tag{3.5}$$

Integrating the (3.5) with respect to $'\omega'$ over $[0,1]$, then

$$\left|\frac{\Psi(\rho_1+\rho_2-\delta)+\Psi(\rho_1+\rho_2-\gamma)}{2}-\frac{1}{\gamma-\delta}\int_{\rho_1+\rho_2-\gamma}^{\rho_1+\rho_2-\delta}\Psi(u)\mathrm{d}u\right|\le\frac{L|\gamma-\delta|}{3}.$$

This completes the proof. □

Theorem 3.8. *Assume that $\Psi:[\rho_1,\rho_2]\to\mathbb{R}$ be a Lipschitzian mapping with $L>0$, then the following inequality holds:*

$$|Z(\rho_1,\rho_2,\delta,\gamma)|\le\frac{L(\gamma-\delta)^2}{32}.$$

Proof. From Lemma 2.1, we have

$$Z(\rho_1,\rho_2,\delta,\gamma)=\frac{\gamma-\delta}{8}\int_0^1(1-2\omega)\left[\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)-\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right]\mathrm{d}\omega \tag{3.6}$$

Implementing the modulus property on (3.6) and applying the Lipschitzian property of Ψ', then

$$|Z(\rho_1,\rho_2,\delta,\gamma)|$$
$$\le\frac{\gamma-\delta}{8}\int_0^1|1-2\omega|\left[\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|-\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|\right]\mathrm{d}\omega$$
$$\le\frac{L(\gamma-\delta)}{8}\int_0^1\omega|1-2\omega||\gamma-\delta|\mathrm{d}\omega$$
$$=\frac{L(\gamma-\delta)|\gamma-\delta|}{32}.$$

This completes the proof. □

Theorem 3.9. *Assume that all the conditions of Lemma 2.1 hold. If there exist constants $-\infty<c<C<\infty$ such that $c\le\Psi'(\delta)\le C,\quad\forall\delta\in[\rho_1,\rho_2]$, then*

$$|Z(\rho_1,\rho_2,\delta,\gamma)|\le\frac{(\gamma-\delta)(C-c)}{16}.$$

Proof. Since $\Psi'(\delta)$ is bounded function, then

$$c-\frac{c+C}{2}\le\Psi'(\delta)-\frac{c+C}{2}\le C-\frac{c+C}{2}.$$

This implies that

$$\left|\Psi'(\delta)-\frac{c+C}{2}\right|\le\frac{C-c}{2}. \tag{3.7}$$

Implementing the modulus property on Lemma 2.1 and then utilizing the observation (3.7), then

$$
\begin{aligned}
&\left|z(\rho_1,\rho_2,\delta,\gamma)\right| \\
&\le \frac{\gamma-\delta}{8}\left[\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)-\frac{c+C}{2}\right|\mathrm{d}\omega\right. \\
&\quad \left.+\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)-\frac{c+C}{2}\right|\mathrm{d}\omega\right] \\
&\le \frac{(\gamma-\delta)(C-c)}{8}\left[\int_0^1 |1-2\omega|\mathrm{d}\omega\right] \\
&= \frac{(\gamma-\delta)(C-c)}{16}. \quad \square
\end{aligned}
$$

Theorem 3.10. *Assume that all the conditions of Lemma 2.1 hold. Suppose that Ψ' is bounded on (ρ_1,ρ_2) i.e. $||\Psi'||_\infty = \sup_\infty |\Psi'(\delta)| < \infty$, then*

$$
|Z(\rho_1,\rho_2,\delta,\gamma)| \le \frac{(\gamma-\delta)||\Psi'||_\infty}{8}.
$$

Proof. Implementing the modulus property on Lemma 2.1 and then utilizing the bounding property of $|\Psi'(\delta|$, then

$$
\begin{aligned}
&\left|z(\rho_1,\rho_2,\delta,\gamma)\right| \\
&\le \frac{\gamma-\delta}{8}\left[\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1-\omega}{2}\delta-\frac{1+\omega}{2}\gamma\right)\right|\mathrm{d}\omega\right. \\
&\quad \left.+\int_0^1 |1-2\omega|\left|\Psi'\left(\rho_1+\rho_2-\frac{1+\omega}{2}\delta-\frac{1-\omega}{2}\gamma\right)\right|\mathrm{d}\omega\right] \\
&\le \frac{(\gamma-\delta)||\Psi'||_\infty}{4}\left[\int_0^1 |1-2\omega|\mathrm{d}\omega\right] \\
&= \frac{(\gamma-\delta)||\Psi'||_\infty}{8}. \quad \square
\end{aligned}
$$

Corollary 3.2. *If we substitute $\delta=\rho_1$ and $\gamma=\rho_2$ in Theorem 3.10, then*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[\frac{\Psi(\rho_1)+\Psi(\rho_2)}{2}+\Psi\left(\frac{\rho_1+\rho_2}{2}\right)\right]-\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\Psi(\delta)\mathrm{d}\delta\right| \\
&\le \frac{(\rho_2-\rho_1)||\Psi'||_\infty}{8}.
\end{aligned} \tag{3.8}
$$

4. Applications

4.1. Applications to means

Now we give some applications to special means for positive real numbers. First of all, we recall some already-known concepts.

1. $A(\rho_1,\rho_2)=\frac{\rho_1+\rho_2}{2}$.
2. $A_w(w_1,w_2;\rho_1,\rho_2)=\frac{w_1\rho_1+w_2\rho_2}{w_1+w_2}$.
3. $\check{}_n(\rho_1,\rho_2)=\left[\frac{{\rho_2}^{n+1}-{\rho_1}^{n+1}}{(\rho_2-\rho_1)(n+1)}\right]^{\frac{1}{n}}, n\in\mathbb{Z}-\{0,-1\}$.

Proposition 4.1. *All the conditions of Theorem 3.1 are satisfied, then*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[A\left((\rho_1+\rho_2-\delta)^2,(\rho_1+\rho_2-\gamma)^2\right)+\left(2A(\rho_1,\rho_2)-A(\delta,\gamma)\right)^2\right]-\check{}_2^2\left(\rho_1+\rho_2-\delta,\,\rho_1+\rho_2-\gamma\right)\right| \\
&\le \frac{\gamma-\delta}{2}\left[A(\rho_1,\rho_2)-\frac{A(\delta,\gamma)}{2}\right].
\end{aligned}
$$

Proof. The proof is achieved by applying $\Psi(z)=z^2$ in Theorem 3.1. $\square$

Proposition 4.2. *All the conditions of Theorem 3.2, are satisfied, then*

$$\left|\frac{1}{2}\left[A\left((\rho_1+\rho_2-\delta)^2,(\rho_1+\rho_2-\gamma)^2\right)+\left(2A(\rho_1,\rho_2)-A(\delta,\gamma)\right)^2\right]-\check{}^{2}_{2}\left(\rho_1+\rho_2-\delta,\,\rho_1+\rho_2-\gamma\right)\right|$$
$$\leq \frac{\gamma-\delta}{4}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left[\left(2A\left(|\rho_1|^q,\,|\rho_2|^q\right)-A_w\left(|\delta|^q,|\gamma|^q,\frac{1}{4},\frac{3}{4}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(2A\left(|\rho_1|^q,\,|\rho_2|^q\right)-A_w\left(|\delta|^q,|\gamma|^q,\frac{3}{4},\frac{1}{4}\right)\right)^{\frac{1}{q}}\right].$$

Proof. The proof is achieved by applying $\Psi(z)=z^2$ in Theorem 3.2. □

4.2. Error bounds

This subsequent part is devoted to establishing some new error bounds of Bullen-type quadrature schemes.

Consider a partition $\Theta:\rho_1=\delta_0<\delta_1<\delta_2<....<\delta_i<\delta_{i+1}<...\delta_n=\rho_2$ of the interval $[\rho_1,\rho_2]$, where $[\delta_i,\delta_{i+1}]$ is any arbitrary subset of $[\rho_1,\rho_2]$. Let $h=\delta_{i+1}-\delta_i$.

$$T(\Theta,\Psi)=\sum_{i=0}^{n-1}\frac{(\delta_{i+1}-\delta_i)^2}{2}\left[\frac{\Psi(\delta_i)+\Psi(\delta_{i+1})}{2}+\Psi\left(\frac{\delta_i+\delta_{i+1}}{2}\right)\right]$$
$$\int_{\rho_1}^{\rho_2}\Psi(\delta)\mathrm{d}\delta=T(\Theta,\Psi)+\bar{R}(\Theta,\Psi),$$

where $\bar{R}(\Theta,\Psi)$ is the error terms.

Proposition 4.3. *From the Theorem 3.4, we have*

$$\left|\bar{R}(\Theta,\Psi)\right|\leq\sum_{i=0}^{n-1}\frac{h^2}{8}\left[\frac{2}{p(p+1)}+\frac{|\Psi'(\delta_i)|^q+|\Psi'(\delta_{i+1})|^q}{q}\right].$$

Proof. To achieve the required result, we set $\delta=\rho_1$ and $\gamma=\rho_2$ in Theorem 3.4 and then implementing this result over subinterval $[\delta_i,\delta_{i+1}]$. □

Proposition 4.4. *From the Theorem 3.5, we have*

$$\left|\bar{R}(\Theta,\Psi)\right|$$
$$\leq\sum_{i=0}^{n-1}\frac{h^2}{8}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{2|\Psi'(\delta_i)|^q+|\Psi'(\delta_{i+1})|^q}{6}\right)^{\frac{1}{q}}\left(\frac{|\Psi'(\delta_i)|^q+2|\Psi'(\delta_{i+1})|^q}{6}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{5|\Psi'(\delta_i)|^q+2|\Psi'(\delta_{i+1})|^q}{12}\right)^{\frac{1}{q}}+\left(\frac{2|\Psi'(\delta_i)|^q+5|\Psi'(\delta_{i+1})|^q}{12}\right)^{\frac{1}{q}}\right].$$

Proof. To achieve the required result, we set $\delta=\rho_1$ and $\gamma=\rho_2$ in Theorem 3.5 and then implementing this result over subinterval $[\delta_i,\delta_{i+1}]$. □

Proposition 4.5. *From the Theorem 3.10, we have*

$$\left|\bar{R}(\Theta,\Psi)\right|\leq\sum_{i=0}^{n-1}\frac{h^2}{8}||\Psi'||_\infty.$$

Proof. To achieve the required result, we set $\delta=\rho_1$ and $\gamma=\rho_2$ in Theorem 3.10 and then implementing this result over subinterval $[\delta_i,\delta_{i+1}]$. □

Similarly many other bounds can be computed by implementing the other main results.

4.3. *q*-digamma function

First, we revisit the notion of the q-digamma function and its mathematical representations: Assume that $0<q<1$. The q-digamma function $\chi_q(\mathfrak{u})$ (for further information, refer to [38]) can be expressed as:

$$\chi_q(\mathfrak{u})=-\ln(1-q)+\ln(q)\sum_{i=0}^{\infty}\frac{q^{i+\mathfrak{u}}}{1-q^{i+\mathfrak{u}}}$$
$$=-\ln(1-q)+\ln(q)\sum_{i=0}^{\infty}\frac{q^{iu}}{1-q^{iu}}.$$

If $q > 1$ and $\mathfrak{u} > 0$, the q-digamma function χ_q can be represented as:

$$\chi_q(\mathfrak{u}) = -\ln(q-1) + \ln(q)\left[\mathfrak{u} - \frac{1}{2} - \sum_{i=0}^{\infty} \frac{q^{-(i+\mathfrak{u})}}{1-q^{-(i+\mathfrak{u})}}\right]$$
$$= -\ln(q-1) + \ln(q)\left[\mathfrak{u} - \frac{1}{2} - \sum_{i=0}^{\infty} \frac{q^{-iu}}{1-q^{-iu}}\right].$$

The notion briefed above shows that for $q > 0$, the function $\chi_q'(\mathfrak{u})$ is completely monotonic on the interval $(0, \infty)$, which implies that it is a convex mapping. From these facts, we can formulate the following important findings concerning the q-digamma function.

Proposition 4.6. *From Theorem 3.3, we acquire*

$$\left|\frac{1}{2}\left[\frac{\chi_q'(\rho_1+\rho_2-\delta)+\chi_q'(\rho_1+\rho_2-\gamma)}{2} + \chi_q'\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\right] - \frac{\chi_q(\rho_1+\rho_2-\delta)+\chi_q(\rho_1+\rho_2-\gamma)}{\gamma-\delta}\right|$$
$$\leq \frac{\gamma-\delta}{8}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\chi''_q(\rho_1)|^q+|\chi''_q(\rho_2)|^q}{4} - \frac{1}{8}|\chi''_q(\delta)|^q - \frac{3}{8}|\chi''_q(\gamma)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|\chi''_q(\rho_1)|^q+|\chi''_q(\rho_2)|^q}{4} - \frac{1}{8}|\chi''_q(\delta)|^q - \frac{3}{8}|\chi''_q(\gamma)|^q\right)^{\frac{1}{q}}\right].$$

Proof. If we take $\Psi(\omega) \mapsto \chi_q'(\omega)$, then result follows directly. □

Proposition 4.7. *From Theorem 3.4, we acquire*

$$\left|\frac{1}{2}\left[\frac{\chi_q'(\rho_1+\rho_2-\delta)+\chi_q'(\rho_1+\rho_2-\gamma)}{2} + \chi_q'\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\right] - \frac{\chi_q(\rho_1+\rho_2-\delta)+\chi_q(\rho_1+\rho_2-\gamma)}{\gamma-\delta}\right|$$
$$\leq \frac{(\gamma-\delta)}{8}\left[\frac{2}{p(p+1)} + \frac{2\left(|\chi''_q(\rho_1)|^q+|\chi''_q(\rho_2)|^q\right)}{q} - \frac{|\chi''_q(\delta)|^q+|\chi''_q(\gamma)|^q}{q}\right].$$

Proof. If we take $\Psi(\omega) \mapsto \chi_q'(\omega)$ in Theorem 3.4, then result follows directly. □

4.4. Modified Bessel functions

Let the $\Omega_d : \mathbb{R} \to (0, 1]$ be defined by

$$\Omega_d(v) = 2^d\Gamma(1+d)v^{-\rho_2}I_p(v).$$

For this, we retrospect the representation of modified Bessel functions, which is given as in [39]:

$$\Omega_d(v) = \sum_{u\geq 0} \frac{\left(\frac{v}{2}\right)^{d+2u}}{u!\Gamma(d+u+1)}.$$

The first and nth-order derivative formula's $\Omega_d(v)$ which are given as in [40]:

$$\Omega_d'(v) = \frac{v}{2(1+d)}\Omega_{d+1}(v), \qquad \frac{\partial^n\Omega_d}{\partial^n v} = 2^{n-2d}\sqrt{\pi}v^{d-n}\Gamma(1+d)_2F_3\left(\frac{1+d}{2}, \frac{2+d}{2}; \frac{1+d-n}{2}, \frac{2+d-n}{2}, 1+d; \frac{v^2}{4}\right),$$

where ${}_2F_3(.,.,.)$ is a hypergeometric function and its integral and summation representation are given as:

$${}_2F_3\left(\frac{1+d}{2}, \frac{2+d}{2}; \frac{1+d-n}{2}, \frac{2+d-n}{2}, (1+d); \frac{v^2}{4}\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{1+d}{2}\right)_k\left(\frac{2+d}{2}\right)_k v^{2k}}{\left(\frac{1+d-n}{2}\right)_k\left(\frac{2+d-n}{2}\right)_k(1+d)_k 4^k k!}.$$

Proposition 4.8. *For any* $[\rho_1, \rho_2] \in \mathbb{R}$*, and* $d > -1$ *then*

$$\left|\frac{1}{2}\left[\frac{(\rho_1+\rho_2-\delta)\Omega_{d+1}(\rho_1+\rho_2-\delta)+(\rho_1+\rho_2-\gamma)\Omega_{d+1}(\rho_1+\rho_2-\gamma)}{4(1+d)} + \frac{\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\Omega_{d+1}\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)}{2(1+d)}\right]\right.$$
$$\left.-\frac{\Omega_d(\rho_1+\rho_2-\delta)+\Omega_d(\rho_1+\rho_2-\gamma)}{\gamma-\delta}\right|$$

$$\leq \frac{(\gamma-\delta)}{8}2^{2-2d}\sqrt{\pi}\Gamma(1+d)\left[|{\rho_1}^{d-2}|\left|{}_2F_3\left(\frac{1+d}{2},\frac{2+d}{2};\frac{d-1}{2},\frac{d}{2},(1+d);\frac{{\rho_1}^2}{4}\right)\right|\right.$$
$$+|{\rho_2}^{d-2}|\left|{}_2F_3\left(\frac{1+d}{2},\frac{2+d}{2};\frac{d-1}{2},\frac{d}{2},(1+d);\frac{{\rho_2}^2}{4}\right)\right|$$
$$\left.-\frac{|\delta^{d-2}|\left|{}_2F_3\left(\frac{1+d}{2},\frac{2+d}{2};\frac{d-1}{2},\frac{d}{2},(1+d);\frac{\delta^2}{4}\right)\right|+|\gamma^{d-2}|\left|{}_2F_3\left(\frac{1+d}{2},\frac{2+d}{2};\frac{d-1}{2},\frac{d}{2},(1+d);\frac{\gamma^2}{4}\right)\right|}{2}\right].$$

Proof. Considering the Theorem 3.1 and applying $\Psi(\upsilon)=\Omega_d'(\upsilon)$, we conclude our required result. □

Proposition 4.9. *For any $[\rho_1,\rho_2]\in\mathbb{R}$, and $d>-1$ then*

$$\left|\frac{1}{2}\left[\frac{(\rho_1+\rho_2-\delta)\Omega_{d+1}(\rho_1+\rho_2-\delta)+(\rho_1+\rho_2-\gamma)\Omega_{d+1}(\rho_1+\rho_2-\gamma)}{4(1+p)}+\frac{\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)\Omega_{d+1}\left(\rho_1+\rho_2-\frac{\delta+\gamma}{2}\right)}{2(1+p)}\right]\right.$$
$$\left.-\frac{\Omega_d(\rho_1+\rho_2-\delta)+\Omega_d(\rho_1+\rho_2-\gamma)}{\gamma-\delta}\right|$$
$$\leq\frac{L(\gamma-\delta)^2}{32}.$$

Proof. Considering the Theorem 3.8 and applying $\Psi(\upsilon)=\Omega_d'(\upsilon)$, then for $n=2$ we conclude our required result. □

4.5. Iterative methods

In the subsequent portion of the study, we give our results applications in non-linear analysis.
Consider the non-linear equation,

$$\Psi(\delta)=0. \tag{4.1}$$

To compute the zeros of non-linear equations is an intriguing aspect of research. In the recent past, numerous methods have been proposed in the literature. Newton's method is rigorously studied iterative schemes and several other methods have been deduced employing different techniques such as quadrature formulae, Taylor's series, interpolating polynomials and decomposition techniques. The relation between quadrature and iterative schemes has been investigated by S. Weerakoon and T. G. I. Fernando in [41] in association with Newton's indefinite integral expression. Motivated by these works, we give an iterative method of our proposed result as an application. First, we recall Newton's integral representations which are proved in [42] as:

$$\Psi(\delta)=\Psi(\delta_n)+\int_{\delta_n}^{x}\Psi'(\omega)\mathrm{d}\omega. \tag{4.2}$$

The method obtained here coincides with schemes proved by G. Nedzhibov [43].

Proposition 4.10. *For any $[\rho_1,\rho_2]\subset\mathbb{R}$ such that $\Psi(\delta)=0$ be a non-linear equation, then*

$$\delta_{n+1}=\delta_n-\frac{\Psi(\delta_n)}{\Psi'(\delta_n)+\Psi'(\gamma_n)+2\Psi'\left(\frac{\delta_n+\gamma_n}{2}\right)},$$

where

$$\gamma_n=\delta_n-\frac{\Psi(\delta_n)}{\Psi'(\delta_n)}.$$

Proof. From (3.8), we have

$$\left|\frac{1}{2}\left[\frac{\Psi(\rho_1)+\Psi(\rho_2)}{2}+\Psi\left(\frac{\rho_1+\rho_2}{2}\right)\right]-\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\Psi(\delta)\mathrm{d}\delta\right|$$
$$\leq\frac{(\rho_2-\rho_1)||\Psi'||_\infty}{8}.$$

This implies that

$$\frac{\rho_2-\rho_1}{2}\left[\frac{\Psi(\rho_1)+\Psi(\rho_2)}{2}+\Psi\left(\frac{\rho_1+\rho_2}{2}\right)\right]+R(\Psi)=\int_{\rho_1}^{\rho_2}\Psi(\delta)\mathrm{d}\delta, \tag{4.3}$$

where

$$R(\Psi) = \frac{(\rho_2 - \rho_1)^2 ||\Psi'||_\infty}{8},$$

remainder of Ψ. Furthermore, we can write

$$\int_{\delta}^{\delta_n} \Psi'(\omega)\mathrm{d}\omega = \frac{(\delta - \delta_n)}{2}\left[\frac{\Psi'(\delta) + \Psi'(\delta_n)}{2} + \Psi'\left(\frac{\delta + \delta_n}{2}\right)\right]. \tag{4.4}$$

Inserting (4.4) in (4.2),

$$\Psi(\delta) = \Psi(\delta_n) + \frac{(\delta - \delta_n)}{2}\left[\frac{\Psi'(\delta) + \Psi'(\delta_n)}{2} + \Psi'\left(\frac{\delta + \delta_n}{2}\right)\right]. \tag{4.5}$$

Making use of (4.1) in (4.5) yields,

$$\delta = \delta_n - \frac{\Psi(\delta_n)}{\Psi'(\delta_n) + \Psi'(\delta) + 2\Psi\left(\frac{\delta+\delta_n}{2}\right)}.$$

This implies that

$$\delta_{n+1} = \delta_n - \frac{\Psi(\delta_n)}{\Psi'(\delta_n) + \Psi'(\gamma_n) + 2\Psi\left(\frac{\gamma_n+\delta_n}{2}\right)},$$

where γ_n is some explicit method. If we take γ_n as the Newton method then we obtain our desired scheme. □

Remark 4.1. The above iterative scheme for finding the solutions of non-linear equations exhibits cubic order of convergence, see [43].

5. Simulations

In the proceeding study segment, we will testify our essential findings through various graphical representations.

Example 5.1. Assume that all the properties of Theorem 3.1 are fulfilled, and considering the mapping $\Psi(u) = \frac{m}{r+2m} u^{\frac{r}{m}+2}$ defined on $\mathbb{R}^+$ with $r \geq 1$ and $m > 1$ be convex functions and $\Psi'(u) = u^{\frac{r}{m}+1}$ with $r \geq 1$ and $m > 1$ be also convex mapping. Fixing the following values $\rho_1 = 1, \delta = 2, \gamma = 3$ and $\rho_2 = 4$, then

$$\left|\frac{m}{4(r+2m)}\left[3^{\frac{r}{m}+2} + 2^{\frac{r}{m}+2} + 2\left(\frac{5}{2}\right)^{\frac{r}{m}+2}\right] - \frac{m^2}{(r+2m)(r+3m)}\left[3^{\frac{r}{m}+3} - 2^{\frac{r}{m}+3}\right]\right|$$
$$\leq \frac{1}{8}\left[1 + 4^{\frac{r}{m}+1} - \frac{2^{\frac{r}{m}+1} + 3^{\frac{r}{m}+1}}{2}\right].$$

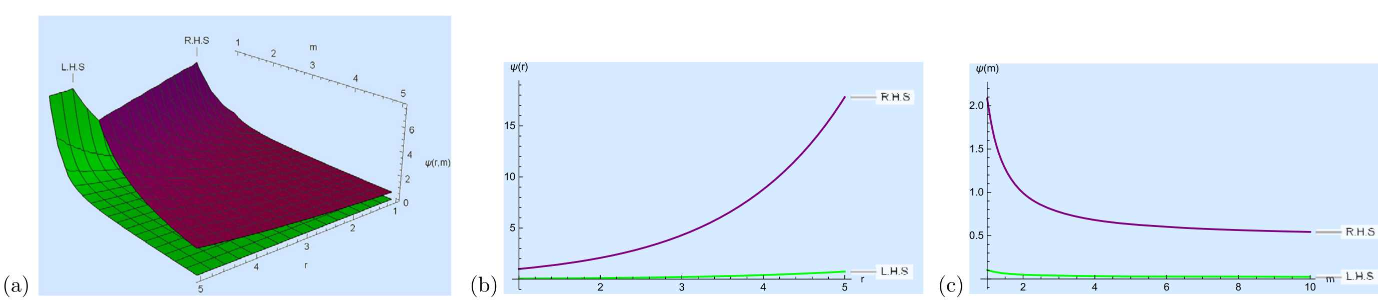


Fig. 5.1. Graphical Illustrations of left (Green) and right (Purple) sides of Theorem 3.1.

- For Fig. 5.1 a, we take $r, m \in [1, 5]$, as variables to plot a graph between the left and right-hand side of Theorem 3.1.
- For Fig. 5.1 b, we take $r \in [1, 5]$, as a variable to plot a graph between the left and right-hand side of Theorem 3.1.
- For Fig. 5.1 c, we take $m \in [1, 10]$, as a variable to plot a graph between the left and right-hand side of Theorem 3.1.

Example 5.2. Assume that all the properties of Theorem 3.2 are fulfilled, and considering the mapping $\Psi(u) = \frac{m}{r+2m} \delta^{\frac{r}{m}+2}$ defined on $\mathbb{R}^+$ with $r \geq 1$ and $m > 1$ be convex functions and $\Psi'(u) = \delta^{\frac{r}{m}+1}$ with $r \geq 1$ and $m > 1$ be also convex mapping. Fixing the following values $\rho_1 = 1, \delta = 2, \gamma = 3$ and $\rho_2 = 4$, then

$$\left|\frac{m}{4(r+2m)}\left[3^{\frac{r}{m}+2} + 2^{\frac{r}{m}+2} + 2\left(\frac{5}{2}\right)^{\frac{r}{m}+2}\right] - \frac{m^2}{(r+2m)(r+3m)}\left[3^{\frac{r}{m}+3} - 2^{\frac{r}{m}+3}\right]\right|$$

$$\leq \frac{1}{8}\left(\frac{1}{3}\right)^{\frac{1}{2}}\left[\sqrt{1+16^{\frac{r}{m}+1}-\frac{4^{\frac{r}{m}+1}+3\times 9^{\frac{r}{m}+1}}{4}}+\sqrt{1+16^{\frac{r}{m}+1}-\frac{3\times 4^{\frac{r}{m}+1}+9^{\frac{r}{m}+1}}{4}}\right].$$

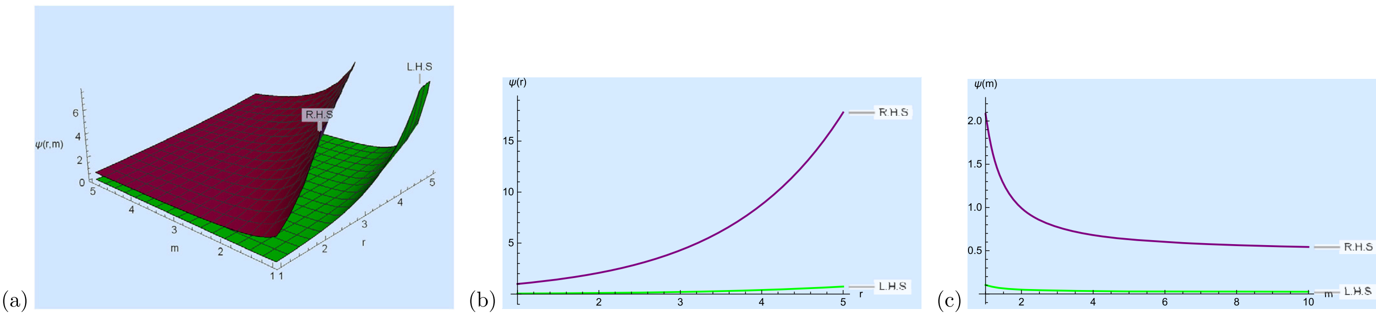


Fig. 5.2. Graphical Illustrations of left (Green) and right (Purple) sides of Theorem 3.2.

- For Fig. 5.2 a, we take $r, m \in [1, 5]$, as variables to plot a graph between the left and right-hand side of Theorem 3.2.
- For Fig. 5.2 b, we take $r \in [1, 5]$, as a variable to plot a graph between the left and right-hand side of Theorem 3.2.
- For Fig. 5.2 c, we take $m \in [1, 10]$, as a variable to plot a graph between the left and right-hand side of Theorem 3.2.

Example 5.3. Assume that all the properties of Theorem 3.3 are fulfilled, and considering the mapping $\Psi(u) = \frac{m}{r+2m}\delta^{\frac{r}{m}+2}$ defined on $\mathbb{R}^+$ with $r \geq 1$ and $m > 1$ be convex functions and $\Psi'(u) = \delta^{\frac{r}{m}+1}$ with $r \geq 1$ and $m > 1$ be also convex mapping. Fixing the following values $\rho_1 = 1, \delta = 2, \gamma = 3$ and $\rho_2 = 4$, then

$$\left|\frac{m}{4(r+2m)}\left[3^{\frac{r}{m}+2}+2^{\frac{r}{m}+2}+2\left(\frac{5}{2}\right)^{\frac{r}{m}+2}\right]-\frac{m^2}{(r+2m)(r+3m)}\left[3^{\frac{r}{m}+3}-2^{\frac{r}{m}+3}\right]\right|$$
$$\leq \frac{1}{8}\left(\frac{1}{2}\right)^{\frac{1}{2}}\left[\sqrt{\frac{1+16^{\frac{r}{m}+1}}{4}-\frac{4^{\frac{r}{m}+1}+3\times 9^{\frac{r}{m}+1}}{8}}+\sqrt{\frac{1+16^{\frac{r}{m}+1}}{4}-\frac{3\times 4^{\frac{r}{m}+1}+9^{\frac{r}{m}+1}}{8}}\right].$$

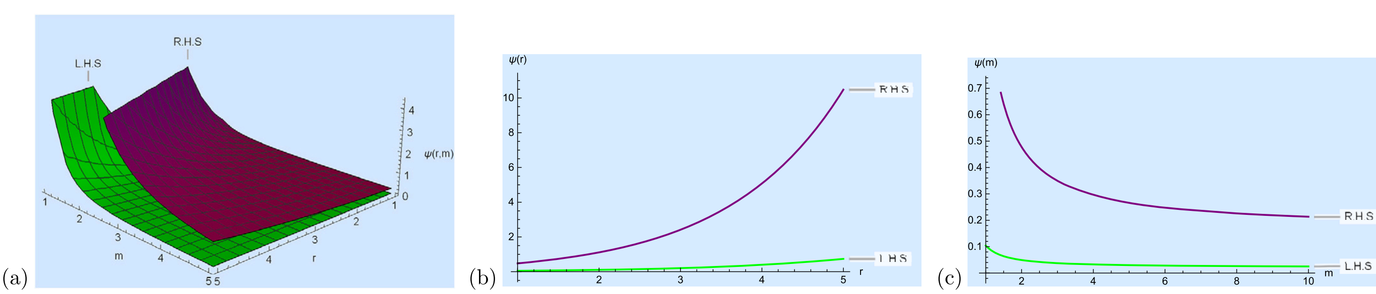


Fig. 5.3. Graphical Illustrations of left (Green) and right (Purple) sides of Theorem 3.3.

- For Fig. 5.3 a, we take $r, m \in [1, 5]$, as variables to plot a graph between the left and right-hand side of Theorem 3.3.
- For Fig. 5.3 b, we take $r \in [1, 5]$, as a variable to plot a graph between the left and right-hand side of Theorem 3.3.
- For Fig. 5.3 c, we take $m \in [1, 10]$, as a variable to plot a graph between the left and right-hand side of Theorem 3.3.

Example 5.4. Assume that all the properties of Theorem 3.4 are fulfilled, and considering the mapping $\Psi(u) = \frac{m}{r+2m}\delta^{\frac{r}{m}+2}$ defined on $\mathbb{R}^+$ with $r \geq 1$ and $m > 1$ be convex functions and $\Psi'(u) = \delta^{\frac{r}{m}+1}$ with $r \geq 1$ and $m > 1$ be also convex mapping. Fixing the following values $\rho_1 = 1, \delta = 2, \gamma = 3$ and $\rho_2 = 4$, then

$$\left|\frac{m}{4(r+2m)}\left[3^{\frac{r}{m}+2}+2^{\frac{r}{m}+2}+2\left(\frac{5}{2}\right)^{\frac{r}{m}+2}\right]-\frac{m^2}{(r+2m)(r+3m)}\left[3^{\frac{r}{m}+3}-2^{\frac{r}{m}+3}\right]\right|$$
$$\leq \frac{1}{8}\left[\frac{8}{3}+4(1+16^{\frac{r}{m}+1})-2(4^{\frac{r}{m}+1}+9^{\frac{r}{m}+1})\right].$$

.

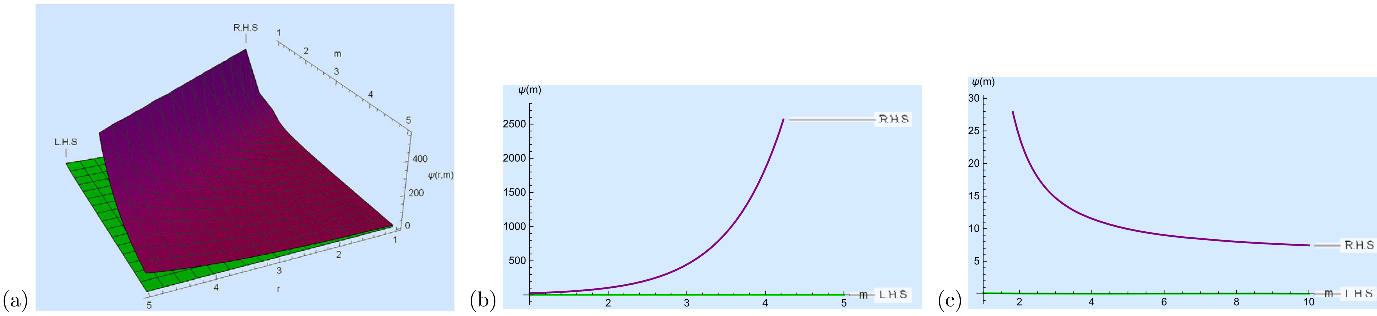


Fig. 5.4. Graphical Illustrations of left (Green) and right (Purple) sides of Theorem 3.4.

- For Fig. 5.4 a, we take $r, m \in [1,5]$, as variables to plot a graph between the left and right-hand side of Theorem 3.4.
- For Fig. 5.4 b, we take $r \in [1,5]$, as a variable to plot a graph between the left and right-hand side of Theorem 3.4.
- For Fig. 5.4 c, we take $m \in [1,10]$, as a variable to plot a graph between the left and right-hand side of Theorem 3.4.

Example 5.5. Assume that all the properties of Theorem 3.5 are fulfilled, and considering the mapping $\Psi(u) = \frac{m}{r+2m}\delta^{\frac{r}{m}+2}$ defined on $\mathbb{R}^+$ with $r \geq 1$ and $m > 1$ be convex functions and $\Psi'(u) = \delta^{\frac{r}{m}+1}$ with $r \geq 1$ and $m > 1$ be also convex mapping. Fixing the following values $\rho_1 = 1, \delta = 2, \gamma = 3$ and $\rho_2 = 4$, then

$$\left|\frac{m}{4(r+2m)}\left[3^{\frac{r}{m}+2} + 2^{\frac{r}{m}+2} + 2\left(\frac{5}{2}\right)^{\frac{r}{m}+2}\right] - \frac{m^2}{(r+2m)(r+3m)}\left[3^{\frac{r}{m}+3} - 2^{\frac{r}{m}+3}\right]\right|$$
$$\leq \frac{1}{8}\left(\frac{1}{3}\right)^{\frac{1}{2}}\left[\sqrt{\frac{1+16^{\frac{r}{m}+1}}{2} - \frac{4^{\frac{r}{m}+1} + 2\times 9^{\frac{r}{m}+1}}{6}} + \sqrt{\frac{1+16^{\frac{r}{m}+1}}{2} - \frac{2\times 4^{\frac{r}{m}+1} + 9^{\frac{r}{m}+1}}{6}}\right.$$
$$\left. + \sqrt{\frac{1+16^{\frac{r}{m}+1}}{2} - \frac{4^{\frac{r}{m}+1} + 5\times 9^{\frac{r}{m}+1}}{12}} + \sqrt{\frac{1+16^{\frac{r}{m}+1}}{2} - \frac{5\times 4^{\frac{r}{m}+1} + 9^{\frac{r}{m}+1}}{12}}\right].$$

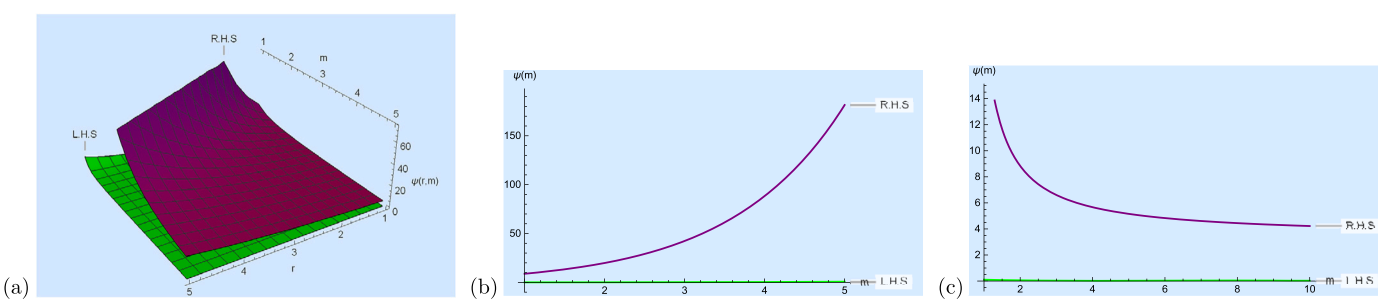
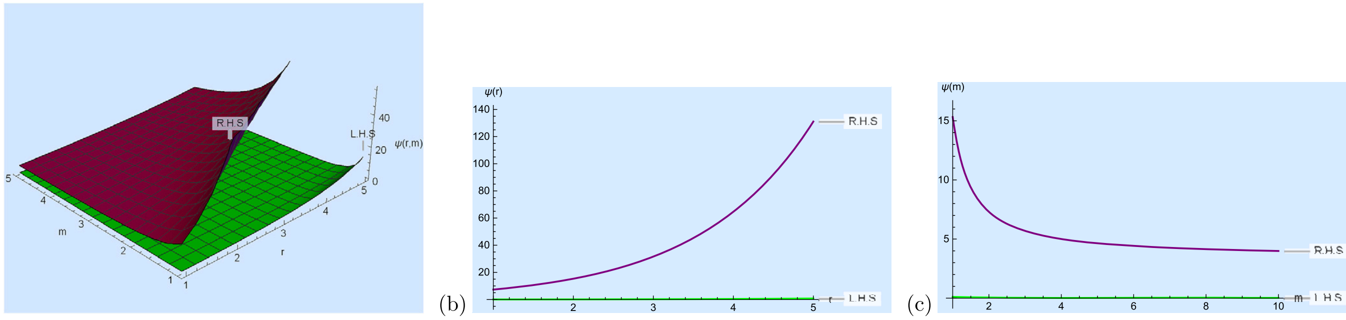


Fig. 5.5. Graphical Illustrations of left(Green) and right (Purple) sides of Theorem 3.5.

- For Fig. 5.5 a, we take $r, m \in [1,5]$, as variables to plot a graph between the left and right-hand side of Theorem 3.5.
- For Fig. 5.5 b, we take $r \in [1,5]$, as a variable to plot a graph between left and right-hand sides of Theorem 3.4.
- For Fig. 5.5 c, we take $m \in [1,10]$, as a variable to plot a graph between the left and right-hand sides of Theorem 3.5.

Example 5.6. Assume that all the properties of Theorem 3.6 are fulfilled, and considering the mapping $\Psi(u) = \frac{m}{r+2m}\delta^{\frac{r}{m}+2}$ defined on $\mathbb{R}^+$ with $r \geq 1$ and $m > 1$ be convex functions and $\Psi'(u) = \delta^{\frac{r}{m}+1}$ with $r \geq 1$ and $m > 1$ be also convex mapping. Fixing the following values $\rho_1 = 1, \delta = 2, \gamma = 3$ and $\rho_2 = 4$, then

$$\left|\frac{m}{4(r+2m)}\left[3^{\frac{r}{m}+2} + 2^{\frac{r}{m}+2} + 2\left(\frac{5}{2}\right)^{\frac{r}{m}+2}\right] - \frac{m^2}{(r+2m)(r+3m)}\left[3^{\frac{r}{m}+3} - 2^{\frac{r}{m}+3}\right]\right|$$
$$\leq \frac{1}{8}\left(\frac{1}{4}\right)^{\frac{1}{2}}\left[\sqrt{\frac{1+16^{\frac{r}{m}+1}}{4} - \frac{3\times 4^{\frac{r}{m}+1} + 5\times 9^{\frac{r}{m}+1}}{32}} + \sqrt{\frac{1+16^{\frac{r}{m}+1}}{4} - \frac{5\times 4^{\frac{r}{m}+1} + 3\times 9^{\frac{r}{m}+1}}{6}}\right.$$
$$\left. + \sqrt{\frac{1+16^{\frac{r}{m}+1}}{4} - \frac{4^{\frac{r}{m}+1} + 7\times 9^{\frac{r}{m}+1}}{32}} + \sqrt{\frac{1+16^{\frac{r}{m}+1}}{4} - \frac{7\times 4^{\frac{r}{m}+1} + 9^{\frac{r}{m}+1}}{32}}\right].$$



(a) (b) (c)

Fig. 5.6. Graphical Illustrations of left (Green) and right (Purple) sides of Theorem 3.6.

- For Fig. 5.6 a, we take $r, m \in [1, 5]$, as variables to plot a graph between the left and right-hand side of Theorem 3.6.
- For Fig. 5.6 b, we take $r \in [1, 5]$, as a variable to plot a graph between the left and right-hand side of Theorem 3.6.
- For Fig. 5.6 c, we take $m \in [1, 10]$, as a variable to plot a graph between the left and right-hand sides of Theorem 3.6.

6. Conclusion

The study of integral inequalities is vitally significant due to various factors. Recent research has witnessed the adaptation of different methodologies to investigate new improvements to previously studied results, including Hermite-Hadamard inequality, which is prosecuted via several techniques. In our article, we have introduced new variants of Hermite-Hadamard-like inequalities which are known as Bullen-Mercer inequalities incorporating the Mercer inequalities. By utilizing the convexity property of the functions, mapping, and more, we have derived numerous upper bounds of Bullen's inequality. Several interesting applications to means and numerical analysis have also been presented. Also, we have validated our main outcomes with the help of a graphical analysis. In the future, we will consider these inequalities in the setting of quantum calculus, fractional calculus, time scale calculus, and majorization theory. Also, these kinds of inequalities can be extended for non-convex mappings like harmonic convexity, p-convexity, h-convexity, and η-convexity. Moreover, iterative methods can be obtained from various integral inequalities and their order of convergence can also be increased. To the best of our knowledge, this is the first attempt to obtain the Bullen-Mercer type of inequality. We hope that the ideas of this paper will inspire further research in this direction.

Declaration of competing interest

The authors declare that they have no conflict of interest.

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References

[1] A.W. Roberts, Convex functions, in: Handbook of Convex Geometry, North-Holland, 1993, pp. 1081–1104.
[2] S.S. Dragomir, C. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, Science direct working paper, (S1574-0358), 04, 2003.
[3] P. Agarwal, S.S. Dragomir, M. Jleli, B. Samet, Advances in Mathematical Inequalities and Applications, Springer, Singapore, 2018.
[4] A.M. Mercer, A variant of Jensen's inequality, J. Inequal. Pure Appl. Math. 4 (4) (2003) 73.
[5] P.S. Bullen, Error estimates for some elementary quadrature rules. Publikacije Elektrotehnickog fakulteta, Ser. Mat. Fiz. 602 (633) (1978) 97–103.
[6] M. Cakmak, Refinements of Bullen-type inequalities for different kind of convex functions via Riemann-Liouville fractional integrals involving Gauss hypergeometric function, Gen. Math. 41 (2018).
[7] M. Cakmak, Some Bullen-type inequalities for conformable fractional integrals, Gen. Math. 28 (2) (2020) 3–17.
[8] S. Erden, M.Z. Sarikaya, Generalized Bullen type inequalities for local fractional integrals and its applications, RGMIA Res. Rep. Collect. 18 (2015) 81.
[9] T. Du, C. Luo, Z. Cao, On the Bullen-type inequalities via generalized fractional integrals and their applications, Fractals 29 (07) (2021) 2150188.
[10] D. Zhao, M.A. Ali, H. Budak, Z.Y. He, Some Bullen type inequalities for generalized fractional integrals, Fractals (2023) 2340060.
[11] F. Hezenci, H. Budak, H. Kara, A study on conformable fractional version of Bullen-type inequalities, Turk. J. Math. 47 (4) (2023) 1306–1317.
[12] H. Boulares, B. Meftah, A. Moumen, R. Shafqat, H. Saber, T. Alraqad, E.E. Ali, Fractional multiplicative Bullen-type inequalities for multiplicative differentiable functions, Symmetry 15 (2) (2023) 451.
[13] P. Agarwal, M. Jleli, M. Tomar, Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals, J. Inequal. Appl. 2017 (1) (2017) 1–10.
[14] P. Agarwal, Some inequalities involving Hadamard-type k-fractional integral operators, Math. Methods Appl. Sci. 40 (11) (2017) 3882–3891.
[15] M.A. Ali, M. Abbas, H. Budak, P. Agarwal, G. Murtaza, Y.M. Chu, New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions, Adv. Differ. Equ. 2021 (2021) 1–21.
[16] K. Mehrez, P. Agarwal, New Hermite-Hadamard type integral inequalities for convex functions and their applications, J. Comput. Appl. Math. 350 (2019) 274–285.
[17] A. Fahad, S.I. Butt, B. Bayraktar, M. Anwar, Y. Wang, Some new Bullen-type inequalities obtained via fractional integral operators, Axioms 12 (7) (2023) 691.
[18] S. Hussain, S. Rafeeq, Y.M. Chu, S. Khalid, S. Saleem, On some new generalized fractional Bullen-type inequalities with applications, J. Inequal. Appl. 2022 (1) (2022) 1–23.
[19] H. Kavurmaci-Onalan, A.O. Akdemir, H. Dutta, Inequalities of Bullen's type for logarithmically convexity with numerical applications, in: 4th International Conference on Computational Mathematics and Engineering Sciences (CMES-2019) 4, Springer International Publishing, 2020, pp. 248–255.
[20] I. Iscan, New refinements for integral and sum forms of Holder inequality, J. Inequal. Appl. 2019 (1) (2019) 304.
[21] M. Kadakal, I. Iscan, H. Kadakal, K. Bekar, On improvements of some integral inequalities, Honam Math. J. 2021 (43) (2021) 441–452.
[22] H. Ogulmus, Z.M. Sarikay, Hermite-Hadamard-Mercer type inequalities for fractional integrals, Filomat 35 (7) (2021) 2425–2436.
[23] I. Iscan, Weighted Hermite-Hadamard-Mercer type inequalities for convex functions, Numer. Methods Partial Differ. Equ. 37 (1) (2021) 118–130.
[24] X. You, M.A. Ali, H. Budak, J. Reunsumrit, T. Sitthi wirattham, Hermite-Hadamard-Mercer-type inequalities for harmonically convex mappings, Mathematics 9 (20) (2021) 2556.

[25] M. Vivas-Cortez, M.U. Awan, M.Z. Javed, A. Kashuri, M.A. Noor, K.I. Noor, A. Vlora, Some new generalized k-fractional Hermite-Hadamard-Mercer type integral inequalities and their applications, AIMS Math. 7 (2022) 3203–3220.
[26] S.I. Butt, A. Nosheen, J. Nasir, K.A. Khan, R. Matendo Mabela, New fractional Mercer-Ostrowski type inequalities with respect to monotone function, Math. Probl. Eng. 2022 (2022).
[27] S. Faisal, M.A. Khan, S. Iqbal, Generalized Hermite-Hadamard-Mercer type inequalities via majorization, Filomat 36 (2) (2022) 469–483.
[28] B. Bin-Mohsin, M.Z. Javed, M.U. Awan, M.V. Mihai, H. Budak, A.G. Khan, M.A. Noor, Jensen-Mercer type inequalities in the setting of fractional calculus with applications, Symmetry 14 (10) (2022) 2187.
[29] J.B. Liu, S.I. Butt, J. Nasir, A. Aslam, A. Fahad, J. Soontharanon, Jensen-Mercer variant of Hermite-Hadamard type inequalities via Atangana-Baleanu fractional operator, AIMS Math. 7 (2) (2022) 2123–2141.
[30] H. Budak, P. Kosem, H. Kara, On new Milne-type inequalities for fractional integrals, J. Inequal. Appl. 2023 (1) (2023) 1–15.
[31] B. Meftah, A. Lakhdari, W. Saleh, A. Kiliçman, Some new fractal Milne-type integral inequalities via generalized convexity with applications, Fractal Fract. 7 (2) (2023) 166.
[32] M.A. Ali, Z. Zhang, M. Feckan, On some error bounds for Milne's formula in fractional calculus, Mathematics 11 (1) (2023) 146.
[33] B. Bin-Mohsin, M.Z. Javed, M.U. Awan, A.G. Khan, C. Cesarano, M.A. Noor, Exploration of quantum Milne-Mercer-type inequalities with applications, Symmetry 15 (5) (2023) 1096.
[34] I.B. Sial, N. Patanarapeelert, M.A. Ali, H. Budak, T. Sitthiwirattham, On some new Ostrowski-Mercer-type inequalities for differentiable functions, Axioms 11 (3) (2022) 132.
[35] M. Vivas-Cortez, M.U. Awan, S. Rafique, M. Zakria, A. Vlora, Some novel inequalities involving Atangana-Baleanu fractional integral operators and applications, AIMS Math. 7 (7) (2022) 12203–12226.
[36] K. Nonlaopon, M.U. Awan, U. Asif, M.Z. Javed, I. Slimane, A. Kashuri, Fractional Jensen-Mercer type inequalities involving generalized Raina's function and applications, Symmetry 14 (10) (2022) 2204.
[37] H.R. Hwang, K.L. Tseng, K.C. Hsu, New inequalities for fractional integrals and their applications, Turk. J. Math. 40 (3) (2016) 471–486.
[38] R. Askey, The q-gamma and q-beta functions, Appl. Anal. 8 (2) (1978/79) 125–141.
[39] G.N. Watson, A Treatise on the Theory of Bessel Functions (vol. 2), The University Press, 1922.
[40] Y.L. Luke (Ed.), Special Functions and Their Approximations, vol. 2, Academic Press, 1969.
[41] S. Weerakoon, T. Fernando, A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett. 13 (8) (2000) 87–93.
[42] J.E. Dennis Jr, R.B. Schnabel, Numerical Methods for Unconstrained Optimization and Nonlinear Equations, Society for Industrial and Applied Mathematics, 1996.
[43] G. Nedzhibov, On a few iterative methods for solving nonlinear equations, Appl. Math. Eng. Econ. 28 (2002) 1–8.