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A DSGE Consumption Function in a CGE Model: Parameter Estimation by CGE Simulation*

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DSGE models incorporate attractive theoretical specifications of the behaviour of forward-looking consumers facing an uncertain future. Central to these specifications is the idea that consuming agents decide their consumption level in year t *by applying a function (policy rule) whose arguments represent information available in year* t*. Using the insight that, under certain conditions, the policy rule (but not the resulting policy) is invariant through time, DSGE modellers have developed the perturbation and other methods for quantitatively specifying policy rules. They have applied these methods in models with limited sectoral disaggregation. In this paper we adapt the perturbation method so that it can be used to specify a policy rule for consumption in a full-scale CGE model. A novel feature of our method is the use of specially constructed CGE simulations to reveal key parameters used in deriving the policy rule. We apply our method in illustrative simulations of the effects of a technology shock in a 70-sector version of the USAGE model of the US economy.*

I Introduction

In the dynamic stochastic general equilibrium (DSGE) theory of consumption, the consuming agent (household, government or a combination of both) determines consumption in year t by applying a rule (the policy function) that takes account of all available information. The information set includes the agent's current wealth and how future wealth will be affected by current consumption. It also includes current values of variables outside the agent's control such as technology and the terms of trade. While the agent does not know the values of future variables with certainty, it does know that it will be applying the same policy function in future years as in the current year. With a steady-growth baseline, this invariance of the policy function allows us to deduce the derivatives of the policy function with respect to wealth and variables exogenous to the agent. From there, we can obtain a first-order approximation to the consumption

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function, showing how aggregate consumption deviates from its baseline path in response to changes in wealth, and changes in variables exogenous to the agent.

DSGE specifications of consumer behaviour are common in macro models. They also appear in models that identify a few broad sectors – see for example, Hornstein and Praschnik (1997; two sectors), Bouakez *et al*. (2009; six sectors) and Rees *et al*. (2016; three sectors). By contrast with DSGE models, computable general equilibrium (CGE) models often encompass enormous amounts of detail. For example, the Global Trade Analysis Project (GTAP) model[1] identifies 65 industries in up to 140 countries together with comprehensive specifications of taxes, trade and environmental variables. With this level of detail, GTAP is the go-to model for thousands of researchers concerned with trade agreements, climate change, international conflict and many other issues. GTAP can be run in various modes: comparative static, recursive dynamic, and dynamic with forward-looking expectations. However, what CGE models such as GTAP do not embrace is uncertainty. Decision-making under certainty by agents in CGE models versus the central role of decision-making under uncertainty in DSGE models is perhaps the clearest demarcation between the two classes of models.

In this paper, we show how a DSGE specification of consumer behaviour can be formulated and applied in a disaggregated CGE model. Our method is a variation of the DSGE perturbation approach (see Schmitt-Grohé & Uribe, 2004; Villaverde *et al*., 2016). However, we rely on CGE simulations to derive the elasticities of wealth at the start of year $t+1$ with respect to wealth at the start of year t, consumption in year t and exogenous variables in year t. These elasticities then become the core ingredients in formulas for the derivatives of the agent's policy function.

Section II sets out a DSGE theory of consumption. It contains the standard ingredients: a wealth-accumulation function showing how wealth in year $t+1$ depends on wealth in year t, consumption in year t and other variables; a specification of lifetime welfare as the sum of current utility and expected welfare beyond the current year; equations flowing from optimising behaviour; and a policy rule. While this theory is broadly standard, it incorporates two non-standard features: the inclusion of current wealth, along with consumption, as an argument of current utility; and the formulation of the policy rule as a function for determining the appropriate value for an extra unit of wealth, rather than as a function for determining consumption directly.

In the context of the DSGE theory from Section II, Section III explains the perturbation method for determining the elasticities of the policy function in the neighbourhood of a non-stochastic, no-growth baseline. We derive formulas for the elasticities of the policy function in terms of first- and second-order elasticities of the wealth-accumulation function. Then we introduce a non-stochastic, steady-growth baseline.

Section IV explains how a CGE model can be used to estimate first-order and second-order elasticities of the wealth-accumulation function. The first-order elasticities refer to the sensitivity of wealth in year $t+1$ to variations in wealth in year t, consumption in year t and other variables in year t. The second-order elasticities refer to the sensitivity of the first-order elasticities to variations in wealth in year t, consumption in year t and other variables in year t.

Section V applies the theory from the previous sections to derive a DSGE consumption function for a 70-sector version of the USAGE CGE model of the US economy. This requires assigning values to all the parameters and coefficients in the DSGE consumption specification. These include the elasticities of the wealth-accumulation function obtained by CGE simulation. Because elasticities obtained in this way are unfamiliar, we make a considerable effort to explain and justify their values.

Section VI presents two sets of USAGE results for the effects of a temporary primary-factor-saving technical change. In one set, the model has our DSGE consumption function in place. In the other set, it has a standard CGE consumption function that relates

[1] The original documentation of this model is Hertel (1997). For more recent accounts, see Corong *et al*. (2017) and Aguiar *et al*. (2019).

current consumption to current disposable income. Comparison of the two sets of results shows how the DSGE specification leads to consumption effects from a temporary shock in year t being spread over future years.

Section VII investigates the sensitivity of the DSGE results to changes in key parameters.

Concluding remarks are in Section VIII.

II A DSGE Consumption Model

This section sets out the DSGE consumption model in a way that will lead to equations suitable for incorporation in a CGE model.

(i) Wealth Accumulation

We assume that the consuming agent accumulates wealth according to an equation of the form:

$$X_{t+1} = J(X_t, Z_t, C_t). \quad (1)$$

In Equation (1) the agent's wealth at the start of year $t+1$ (X_{t+1}) is a function of wealth at the start of year t (X_t), consumption during year t (C_t) and variables exogenous to the agent (Z_t). These exogenous variables could include technologies, consumer preferences, the terms of trade, and the state of business confidence. They can be thought of as influencing components of the agent's income such as wage rates, employment and profits.

We assume that the Z vector for year t is known to the consuming agent in year t, but the Z vectors for future years are known in year t only in a probabilistic form. For most of this paper, we assume away shock persistence: that is, we assume that the outcome for Z in $t+1$ is independent of the outcome in t. In Section VII, we introduce shock persistence in sensitivity simulations. In what follows, we indicate uncertainty by σ. We have in mind a variance–covariance matrix for the components of Z.

(ii) Expected Lifetime Welfare

We specify the consuming agent's expectation held in year t of its lifetime welfare

$$V(X_t, Z_t, \sigma) = U(C_t, X_t) + \beta E_t[V(X_{t+1}, Z_{t+1}, \sigma)]. \quad (2)$$

Apart from one feature, to be discussed shortly, specification (2) is standard in the DSGE literature. Lifetime welfare expected in year t depends on the data available to the agent in year t: current wealth, X_t; current values of exogenous variables, Z_t; and the probabilistic generating process for the Zs represented here simply as σ. Lifetime welfare expected in year t can be divided into two parts. The first part is utility derived in year t. The second part is the discounted expectation held in year t concerning lifetime welfare that will be expected in year $t+1$. The discount factor is the parameter β whose value is between 0 and 1. Expectations held in year t concerning unknown values of future variables are indicated by E_t.

The non-standard feature is the inclusion in the utility contribution for year t of current wealth, X_t, not just consumption C_t. We assume that wealth contributes to lifetime welfare not only by facilitating future consumption but also by providing a sense of security in each year. For concreteness, we give U the specific form

$$U(t) = \frac{\left(X_t^{1-\theta} C_t^{\theta}\right)^{1-\gamma}}{1-\gamma}, \quad (3)$$

where γ and θ are positive parameters, with $\gamma \neq 1$ and $\theta \leq 1$; and $U(t)$ is the value of the U function when its arguments have their year t values.

In Section III we will find that the method being pursued in this paper produces a consumption function that gives zero responses to variations in σ (see Equation 31). Thus, the inclusion of X_t in the utility function seems a potentially attractive option for capturing effects of uncertainty. For example, we might simulate growing uncertainty by decreasing θ. In Section V, we find that the inclusion of wealth has another useful role: it facilitates the calibration of our DSGE consumption function to the data in our CGE model. With θ set to 1, which excludes wealth from $U(t)$, the calibrated value of β is greater than 1. A value of θ less than 1 is required to produce a calibrated value for β in the theoretically permissible range. In Section VII we conduct sensitivity simulations to reveal effects of varying θ.

(iii) Optimising Behaviour

Optimising consumption between the current year and the future requires

$$U_C(t) + \beta\{E_t[V_X(t+1)]\}\frac{\partial X_{t+1}}{\partial C_t} = 0, \quad (4)$$

where $U_C(t)$ is the partial derivative of $U(t)$ with respect to C_t, calculated in (3) holding X_t constant; $V_X(t)$ is the partial derivative of V with respect to X_t, calculated holding Z_t and σ constant; and $\partial X_{t+1}/\partial C_t$ is the partial derivative of X_{t+1} with respect to C_t, calculated in (1) with X_t and Z_t held constant.

In Equation (4), the effect on utility in year t from an extra unit of consumption is matched with the expected effect on lifetime welfare of the reduction in wealth at the start of $t+1$ resulting from the extra consumption in year t.

Differentiating the V function in (2) with respect to X_t gives

$$V_X(t) = U_X(t) + U_C(t)\frac{\partial C_t}{\partial X_t} + \beta E_t\left[V_X(t+1)\frac{\partial X_{t+1}}{\partial X_t} + V_X(t+1)\frac{\partial X_{t+1}}{\partial C_t}\frac{\partial C_t}{\partial X_t}\right], \quad (5)$$

where $U_X(t)$ is the partial derivative of $U(t)$ with respect to X_t, calculated in (3) holding C_t constant; $\partial X_{t+1}/\partial X_t$ is the partial derivative of X_{t+1} with respect to X_t, calculated in (1) with C_t and Z_t held constant; and $\partial C_t/\partial X_t$ gives the effect on consumption of a unit increase in X_t with Z_t and σ held constant.

Consistent with the envelope theorem, we can use (4) to reduce (5) to

$$V_X(t) = U_X(t) + \beta\{E_t[V_X(t+1)]\}\frac{\partial X_{t+1}}{\partial X_t}, \quad (6)$$

where the value of having an extra unit of wealth at the start of year t is calculated as the effect on utility in year t from simply possessing extra wealth *plus* the expected effect on lifetime welfare from the potential[2] increase in wealth at the start of year $t+1$ resulting from the extra unit of wealth at the start of year t.

(iv) The Policy Rule

We introduce the consuming agent's decision strategy for year t by a policy rule specified as

$$V_X(t) = M[X_t, Z_t, \sigma]. \quad (7)$$

We assume that the agent expects in year t to be implementing the same policy rule in year $t+1$:

$$E_t V_X(t+1) = E_t M[X_{t+1}, Z_{t+1}, \sigma]. \quad (8)$$

In most expositions of DSGE theory, the policy rule expresses consumption in year t, C_t, as a function of data available in year t. Instead of an equation such as (6) motivating the policy rule, in standard expositions it is motivated by an equation that expresses C_t as the expectation held in year t for the value of a function that includes consumption in year $t+1$ and other variables for year $t+1$. This is a consumption-focused Euler equation.[3] In the standard approach, $V_X(t)$ and $E_t[V_X(t+1)]$ are eliminated from the model. Performing the eliminations and deriving the consumption-focused Euler equation is usually straightforward in models in which there is only one predetermined endogenous variable (X_t is a scalar). However, in models in which there are multiple predetermined endogenous variables (X_t is a vector), elimination of $V_X(t)$

[2] The consumer need not carry all the extra wealth into year $t+1$. The envelope theorem means that the welfare effect of extra wealth at the start of year t does not depend on the way it is allocated between consumption in year t and extra wealth accumulation.

[3] For example, in the neoclassical model in which capital is the only predetermined endogenous variable, the Euler equation can be written as $C_t^{-\gamma} = \beta E_t\left[C_{t+1}^{-\gamma}\left((1-\delta) + A_{t+1}\alpha K_{t+1}^{\alpha-1}\right)\right]$, where C_t is consumption in year t, A_{t+1} and K_{t+1} are productivity and capital in year $t+1$, and β, α and δ are parameters.

and $E_t[V_X(t+1)]$, now vectors, may not be directly possible. In any case, under our method it is not necessary.[4]

III Linearising the Equations of the DSGE Consumption Model and Deriving the Derivatives of the Policy Function

By applying the perturbation method we can derive formulas though which the derivatives of the M function can be evaluated in the vicinity of a non-stochastic steady-state baseline solution of our model. This is a solution in which $\sigma = 0$ and the year $t+1$ values of all variables are the same as the year t values.

To apply the method we need versions of Equations (1), (4), (6), (7) and (8) linearised around a non-stochastic steady state. CGE modellers are accustomed to specifying equations in terms of elasticities and percentage changes in variables rather than derivatives and changes in variables. Consequently, to ease the transfer of DSGE specifications into CGE modelling it is useful to express DSGE linearised equations mainly in elasticity percentage-change form.

We derive the linearised system in a general form and use the general form in two tasks: first, to determine the elasticity (M_X) of the policy rule with respect to wealth (X_t); and second, to specify the consumption function, that is the function relating C_t to X_t and Z_t.

To derive the linearised system we start by writing the linearised version of (1) as

$$x_{t+1} = J_X x_t - J_C c_t + J_Z z_t, \tag{9}$$

where x_{t+1}, x_t, c_t and z_t are *percentage* deviations in X_{t+1}, X_t, C_t and Z_t from their steady-state values; and J_X, J_C and J_Z (without t arguments) are elasticities of the J function in (1) evaluated at steady-state values of the variables. J_X and J_C are scalars and J_Z is a vector of length r where r is the number of exogenous variables. In general, the elasticities are defined by:

$$J_X(t) = \frac{\partial X_{t+1}}{\partial X_t}\frac{X_t}{X_{t+1}}, \tag{10}$$

$$J_C(t) = -\frac{\partial X_{t+1}}{\partial C_t}\frac{C_t}{X_{t+1}}, \tag{11}$$

$$J_Z(t) = \left(\frac{\partial X_{t+1}}{\partial Z_{1t}} * \frac{Z_{1t}}{X_{t+1}}, \ \ldots, \ \frac{\partial X_{t+1}}{\partial Z_{rt}} * \frac{Z_{rt}}{X_{t+1}}\right). \tag{12}$$

In Equations (10)–(12) we use the notation $J_X(t)$, $J_C(t)$ and $J_Z(t)$ to denote elasticities evaluated with variables set at their year t values. We assume that both $J_X(t)$ and $J_C(t)$ are non-negative. As will shortly become apparent, we will need to compute percentage changes in $J_X(t)$ and $J_C(t)$. Percentage changes in negative quantities are not well defined. $J_X(t)$ presents no problem. We expect the elasticity of future wealth with respect to current wealth to be positive. However, we expect an increase in current consumption to reduce future wealth. Consequently, to ensure that J_C is positive, we define it in (11) with a negative sign on the right-hand side. The same problem does not arise with $J_Z(t)$. Its components can be either positive or negative. We do not need to compute percentage changes in Z elasticities.

Next, we use (3), (8) and (11) in (4) to obtain

$$\theta C_t^{\theta-1} X_t^{1-\theta}\left(X_t^{1-\theta} C_t^{\theta}\right)^{-\gamma} = \beta \tilde{M}(t) J_C(t) \frac{X_{t+1}}{C_t}, \tag{13}$$

where

$$\tilde{M}(t) = E_t[M(t+1)]. \tag{14}$$

Linearising (13) around the steady-state baseline gives

$$\begin{aligned}(\theta - 1 - \theta\gamma)c_t + (1-\theta)(1-\gamma)x_t \\ = \tilde{m}(t) + j_C(t) + x_{t+1} - c_t.\end{aligned} \tag{15}$$

where $\tilde{m}(t)$ and $j_C(t)$ are percentage deviations in $\tilde{M}(t)$ and $J_C(t)$ from their steady-state values. Notice that by using the negative sign on the right hand side of (11) so that J_C is positive, we ensure the existence of the percentage change $j_C(t)$.

[4] Dixon and Rimmer (2020) provide an example of a model in which there are two predetermined endogenous variables: capital and a lagged wage rate.

We calculate $j_C(t)$ via a linearised version of J_C:

$$j_C(t) = J_{CX}x_t + J_{CC}c_t + J_{CZ}z_t, \tag{16}$$

where J_{CX}, J_{CC} and J_{CZ} (without t arguments) are elasticities of the J_C function in (11) evaluated at steady-state values of the variables. In general, these elasticities are defined by

$$J_{CX}(t) = \frac{\partial J_C(t)}{\partial X_t}\frac{X_t}{J_C(t)}, \tag{17}$$

$$J_{CC}(t) = \frac{\partial J_C(t)}{\partial C_t}\frac{C_t}{J_C(t)}, \tag{18}$$

$$J_{CZ}(t) = \left(\frac{\partial J_C(t)}{\partial Z_{1t}}\frac{Z_{1t}}{J_C(t)}, \ \ldots, \ \frac{\partial J_C(t)}{\partial Z_{rt}}\frac{Z_{rt}}{J_C(t)}\right). \tag{19}$$

Now we work on (6) and use (3), (7), (8), (14) and (10) in (6) to obtain:

$$\begin{aligned} M(t) = {} & (1-\theta)X_t^{-\theta}C_t^{\theta}\left(X_t^{1-\theta}C_t^{\theta}\right)^{-\gamma} \\ & + \beta\{\widetilde{M}(t)\}J_X(t)\frac{X_{t+1}}{X_t}. \end{aligned} \tag{20}$$

Linearising (20) around the steady-state baseline gives.

$$\begin{aligned} Mm(t) = {} & U_X[(\theta\gamma-\theta-\gamma)x_t + (1-\gamma)\theta c_t] \\ & + \beta M J_X[\tilde{m}(t) + j_X(t) + x_{t+1} - x_t], \end{aligned} \tag{21}$$

where $m(t)$ and $j_X(t)$ are percentage deviations in $M(t)$ and $J_X(t)$ from their steady-state values. In deriving (21) we use the facts that the steady-state values of $M(t)$ and $E_t[M(t+1)]$ are the same and can be written as M, and that the steady-state values of X_t and X_{t+1} are the same.

We calculate $j_X(t)$ via a linearised version of J_X:

$$j_X(t) = J_{XX}x_t + J_{XC}c_t + J_{XZ}z_t \tag{22}$$

where J_{XX}, J_{XC} and J_{XZ} (without t arguments) are elasticities of the J_X function in (10) evaluated at steady-state values of the variables. In general, these elasticities are defined by

$$J_{XX}(t) = \frac{\partial J_X(t)}{\partial X_t}\frac{X_t}{J_X(t)}, \tag{23}$$

$$J_{XC}(t) = \frac{\partial J_X(t)}{\partial C_t}\frac{C_t}{J_X(t)}, \tag{24}$$

$$J_{XZ}(t) = \left(\frac{\partial J_X(t)}{\partial Z_{1t}}\frac{Z_{1t}}{J_X(t)}, \ \ldots, \ \frac{\partial J_X(t)}{\partial Z_{rt}}\frac{Z_{rt}}{J_X(t)}\right). \tag{25}$$

We write the linearised forms of (7) and (8) as

$$m(t) = M_X x_t + M_Z z_t + M_\sigma d\sigma \tag{26}$$

and

$$\tilde{m}(t) = M_X x_{t+1} + M_\sigma d\sigma, \tag{27}$$

where $d\sigma$ is the deviation in σ form its steady-state value of zero; M_X and M_Z are the steady-state values of the elasticities of the M function with respect to X and Z; and M_σ is the steady-state value of the semi-elasticity of M with respect to σ. These are defined in general by:

$$\begin{aligned} M_X(t) &= \frac{\partial M(t)}{\partial X_t}\frac{X_t}{M(t)}, \\ M_Z(t) &= \left(\frac{\partial M(t)}{\partial Z_{1t}}\frac{Z_{1t}}{M(t)}, \ \ldots, \ \frac{\partial M(t)}{\partial Z_{rt}}\frac{Z_{rt}}{M(t)}\right), \\ M_\sigma(t) &= \frac{1}{M(t)}\frac{\partial M(t)}{\partial \sigma}. \end{aligned} \tag{28}$$

Four aspects of (26) and (27) need clarification. First, the determination of x_{t+1} follows in a non-stochastic way from the year t deviations in variables from their baseline values. Consequently, the expectation operator is not applied to x_{t+1} in (27). Second, as mentioned in the discussion of Equation (1), we assume here that year t contains no information about exogenous variables in year $t+1$; that is, in year t the agent expects Z_{t+1} to have its steady-state baseline value. This explains the omission

TABLE 1
System for Evaluating Elasticities of M

(9)	$x_{t+1} = J_X x_t - J_C c_t + J_Z z_t$
(15)	$(\theta - 1 - \theta\gamma)c_t + (1-\theta)(1-\gamma)x_t = \tilde{m}(t) + j_C(t) + x_{t+1} - c_t$
(16)	$j_C(t) = J_{CX}x_t + J_{CC}c_t + J_{CZ}z_t$
(21)	$Mm(t) = U_X[(\theta\gamma - \theta - \gamma)x_t + (1-\gamma)\theta c_t] + \beta M J_X[\tilde{m}(t) + j_X(t) + x_{t+1} - x_t]$
(22)	$j_X(t) = J_{XX}x_t + J_{XC}c_t + J_{XZ}z_t$
(26)	$m(t) = M_X x_t + M_Z z_t + M_\sigma d\sigma$
(27)	$\tilde{m}(t) = M_X x_{t+1} + M_\sigma d\sigma$

from (27) of percentage deviations in exogenous variables. When we introduce shock persistence in Section VII, Equation (27) will be modified to include z_t terms. Third, in our linearised system we allow a change in σ, $d\sigma$, to occur in year t and to be viewed as permanent by the consuming agent; that is, in year t the agent expects the year $t+1$ deviation in σ to be the same as the actual deviation in year t. Fourth, we use the change in σ and the semi-elasticity to avoid division by zero.

(i) Evaluating $\mathrm{M_X}$

We work with Equations (9), (15), (16), (21), (22), (26) and (27). These are reproduced for convenience in Table 1. We take the values of the parameters γ, β and θ, as given and we assume that the coefficients J_X, J_C, J_Z, J_{CC}, J_{CX}, J_{CZ}, J_{XC}, J_{XX}, J_{XZ}, M and U_X can be evaluated from the non-stochastic steady-state solution of our model. Sections IV and V explain how these evaluations are achieved.

For evaluating M_X, we treat x_t, z_t, and $d\sigma$ as exogenous variables.

To obtain M_X, we set

$$x_t = 1, z_t = 0 \text{ and } d\sigma = 0. \tag{29}$$

With z and $d\sigma$ set in this way, M_Z and M_σ disappear from the seven equations. This leaves seven unknowns, x_{t+1}, c_t, $j_C(t)$, $j_X(t)$, $m(t)$, $\tilde{m}(t)$ and M_X, in seven equations. As shown in Appendix I, we can solve the seven equations for the seven unknowns. With x_t set to 1, the valid solution for M_X (as explained in Appendix I, there is more than one solution) reveals the steady-state elasticity of $M(t)$ with respect to X_t: it is the percentage effect on M of a 1 per cent increase in X_t, holding all other exogenous variables constant.

(ii) Evaluating $\mathrm{M_\sigma}$

Again we work with the equations in Table 1. We treat M_X as known and x_t, z_t, and $d\sigma$ as exogenous variables. To obtain M_σ we set

$$x_t = 0, z_t = 0 \text{ and } d\sigma = 1. \tag{30}$$

With x_t, z_t and $d\sigma$ set in this way, Table 1 provides seven equations with seven unknowns: x_{t+1}, c_t, $j_C(t)$, $j_X(t)$, $m(t)$, $\tilde{m}(t)$ and M_σ. As shown in Appendix I, solving these equations gives:

$$M_\sigma = 0. \tag{31}$$

This result will be familiar to DSGE modellers from other DSGE formulations; see, for example, Schmitt-Grohé and Uribe (2004). It means that a small increase in uncertainty from the zero level ($\sigma = 0$) has no effect on consumption. This is disappointing. The 'S' in DSGE holds out hope that models in this tradition will help us understand the role of uncertainty in determining macroeconomic aggregates. However, realising this potential requires an nth-order approximation to the policy rule with $n > 1$. Ours is a first-order approximation. In a CGE context with no explicit form for the wealth-accumulation function (Eqn 1), we do not know how to form a higher-order approximation.

(iii) The DSGE Consumption Function

With the values of M_X and M_σ now known, we can use our seven equations to deduce how consumption (C_t) depends on wealth

(X_t) and current values of exogenous variables (Z_t). As shown in Appendix I, we substitute variables and eliminate equations, eventually arriving at

$$c_t = \frac{[(M_X+1)J_X + J_{CX} - (1-\theta)(1-\gamma)]}{[(\theta-\theta\gamma) + (M_X+1)J_C - J_{CC}]} x_t + \frac{[(M_X+1)J_Z + J_{CZ}]}{[(\theta-\theta\gamma) + (M_X+1)J_C - J_{CC}]} z_t. \quad (32)$$

(iv) Moving to a Steady-Growth Baseline

The perturbation method that we have described so far is predicated on no-growth baselines. In these baselines, the value of every variable in year $t+1$ is the same as in year t. However, realism demands that we allow for economic growth. Assume, for example, that we are dealing with an economy such as Australia or the USA in which the investment share in GDP is about 20 per cent. This cannot be reproduced in a no-growth steady state. It is consistent with a situation typical of these countries in which the capital-to-output ratio is relatively stable at around 2.5, the depreciation rate is 5 per cent, and capital and output grow at about 3 per cent ($20 = 2.5(3+5)$). In a no-growth baseline, the investment share of GDP would be unrealistically low.

Although in the DSGE framework we cannot go all the way to a realistic baseline, we can take a step in that direction by introducing steady growth. We do this by adopting a baseline in which the baseline value (denoted by b) of every variable Q can be described by

$$Q_{b,t+1} = Q_{b,t}\xi(Q), \quad (33)$$

where $\xi(Q)$ is the steady-state growth factor (1 plus growth rate) for variable Q.

In the applications described in Sections VI and VII, we set up a CGE model for the USA with a baseline in which each industry increases its output at 3 per cent per year. We do this by assuming: 2 per cent annual labour-saving technical progress in each industry with no other changes in technology; 1 per cent annual growth in aggregate employment; 3 per cent annual outward movement in foreign demand curves for all US exports; no changes in prices of imported products; 3 per cent annual growth in public expenditures; unitary consumer expenditure elasticities for all products; no changes in consumer preferences; and initial investment–capital ratios and depreciation rates implying 3 per cent capital growth. Relative to the assumptions that CGE modellers normally use in baselines, these steady-growth assumptions reduce realism. Unfortunately, this seems to be an unavoidable cost of adopting DSGE theory.

In a steady-growth baseline, growth-discounted (gd) variables exhibit zero growth. A growth-discounted variable is a variable divided by its growth factor:

$$Q_t^{\text{gd}} = \frac{Q_t}{\xi(Q)}. \quad (34)$$

By reinterpreting all the variables in the DSGE consumption specification as growth discounted, we form a system of equations that has a no-growth baseline as part of a larger system that has a steady-growth baseline. With this reinterpretation, the analysis of Table 1 based on a no-growth baseline becomes applicable with a steady-growth baseline. We simply reinterpret c_t, x_t, $m(t)$, etc. as percentage deviations in growth-discounted variables from the values they had on the steady-growth baseline. The elasticities in Table 1 become elasticities of growth-discounted variables with respect to growth-discounted variables evaluated at baseline values of variables. With a steady-growth baseline, the percentage deviations in growth-discounted variables from their baseline values are the same as the percentage deviations in the undiscounted variables from their baseline values, and the elasticities for growth-discounted variables are the same as the corresponding elasticities for undiscounted variables.

While the switch from a no-growth baseline to a steady-growth baseline leaves Table 1 intact with only a reinterpretation of variables, it *does* require adjustment of our underlying economic theory. Now, in Equation (3), growth in consumption and wealth with growth factors $\xi(C)$ and $\xi(X)$ maintains annual utility (U) in year $t+1$ at its year t level. If $\xi(C)$ and $\xi(X)$ are greater than 1, then for maintenance of a given level of annual utility, consumption and wealth must grow. We can interpret this as meaning that maintenance of utility requires consumption and wealth to grow in line with population and community aspirations reflected in normal growth rates in per capita consumption and wealth.

IV Estimation of the Elasticities of the Accumulation Relationship

Implementation of consumption functions such as Equation (32) requires evaluation on a steady-growth baseline of the elasticities, J_X, J_C, J_Z, J_{CC}, J_{CX}, J_{CZ}, J_{XC}, J_{XX} and J_{XZ}. This presents no difficulty when we are dealing with small-scale models in which the accumulation relationship is simple and explicit. But how do we evaluate the elasticities in a large-scale CGE model in which wealth accumulation is not represented by a simple explicit function of X_t, Z_t and C_t, but instead is the outcome of a system of equations involving a large number of variables, including wage rates, profits, taxes, interest rates, capital stocks and employment?

Before we can explain our evaluation method, we need to fill in some background. Simulations with CGE models of the type we use consist of two runs, the baseline and policy runs.[5] Usually the baseline is intended as a business-as-usual, year-on-year picture of the paths for the myriad of variables in CGE models, such as employment and output by industry. CGE modellers often build into the baseline trends in technology, consumer preferences and commodity prices together with demographic projections. The policy run is usually undertaken with a different closure (choice of exogenous variables) from that in the baseline. For example, macro variables in policy runs are normally endogenous, whereas in the baseline they are often exogenous so that the modeller can build into the baseline macro forecasts provided by specialist forecasting groups such as the IMF.

With key exceptions, all of the exogenous variables in the policy run follow the same paths that they had either endogenously or exogenously in the baseline. The key exceptions are usually policy variables. For example, if the purpose of the simulation is to determine the effects of proposed tariff changes, the relevant tariff variables are put on paths in the policy run different from their baseline paths. If none of the exogenous variables in the policy run is moved off its baseline path, then, despite a different closure, the policy run will give the same solution as the baseline run. Consequently, differences between policy and baseline results show the effects of deviations in policy variables (e.g., tariffs) from their baseline paths. Because we are normally interested in macro effects, macro variables must be endogenous in policy runs, although as mentioned earlier, they may be exogenous in the baseline. This is the reason the policy closure is usually different from the baseline closure.

We return now to the problem of estimating J_X, J_C, J_Z, J_{CC}, J_{CX}, J_{CZ}, J_{XC}, J_{XX} and J_{XZ}.

The first step in our evaluation method is to set up the CGE model with a steady-growth baseline. As described in the previous section, we made technology, preference and population assumptions that generated a 3 per cent steady-growth baseline for our US model.

The second step is to estimate the first-order elasticities, J_X, J_C and J_Z by conducting a series of policy runs that generate deviations away from the steady-growth baseline. In these estimating policy runs, consumption in year t (C_t) and the agent's wealth at the start of year t (X_t) are exogenous, together with the naturally exogenous variables (Z_t).

Is this legitimate? Can we evaluate the required elasticities in simulations with a

[5] This includes models such as MONASH, USAGE and VU-NATIONAL; see for example, Dixon and Rimmer (2002) and Dixon *et al*. (2013). These models use GEMPACK software (see Horridge *et al*., 2013, 2018). This software solves models expressed in percentage changes of variables and elasticities. It is ideal for our DSGE method which requires working with derivatives, elasticities and first-order approximations.

model that does not include the DSGE consumption behaviour that we are aiming to add to the model? The exogeneity of C and X in the estimating simulations means that there is no problem. Given the specification in the model of production, trade, taxes, etc., what we need to know for the implementation of Equation (32) is the sensitivity of next-year's wealth to:

- an exogenously imposed 1 per cent increase in this year's wealth, holding consumption and exogenous variables constant;
- an exogenously imposed 1 per cent increase in consumption, holding this year's wealth and exogenous variables constant; and
- exogenously imposed 1 per cent increases in each exogenous variable in turn, holding this year's wealth and consumption constant.It is these sensitivities that are revealed in our estimating simulations.

By imposing a 1 per cent shock on X_t (i.e., moving X_t 1 per cent above its baseline value) while holding C_t and Z_t at their baseline values, we can observe $J_X(\overline{C},\overline{X},\overline{Z})$, where the bars on variables denote growth-discounted values on the steady-growth baseline. This is done by looking at the percentage deviation result (x_{t+1}) for wealth at the start of year $t+1$.

By imposing a 1 per cent shock on C_t (i.e., moving C_t 1 per cent above its baseline value) while holding X_t and Z_t at their baseline values, we can observe $J_C(\overline{C},\overline{X},\overline{Z})$. Again, this is done by looking at the percentage deviation result (x_{t+1}) for wealth at the start of year $t+1$ but reversing its sign. Recall that J_C is the negative of $(\partial X_{t+1}/\partial C_t)(C_t/X_{t+1})$.

By imposing 1 per cent shocks on components of Z_t while holding C_t and X_t at their baseline values, we can observe components of $J_Z(\overline{C},\overline{X},\overline{Z})$.

The final step is to evaluate the second-order elasticities. We start with $J_{XC}(\overline{C},\overline{X},\overline{Z})$. In evaluating this elasticity we use an additional policy simulation in which a 1 per cent shock is applied to X_t but not in the baseline situation. Instead we set $x_t = 1$ in the situation reached in the simulation that revealed $J_C(\overline{C},\overline{X},\overline{Z})$. In this additional simulation, the result for x_{t+1} reveals $J_X(\overline{C}\times 1.01,\overline{X},\overline{Z})$. That is, the additional simulation shows the effect on next year's wealth of a 1 per cent increase in this year's wealth imposed in a situation in which consumption is 1 per cent above baseline and X_t and Z_t are on baseline. Recalling that J_{XC} is the percentage effect on J_X of moving C by 1 per cent, we calculate $J_{XC}(\overline{C},\overline{X},\overline{Z})$ as

$$J_{XC}(\overline{C},\overline{X},\overline{Z}) = 100\frac{J_X(\overline{C}\times 1.01,\overline{X},\overline{Z}) - J_X(\overline{C},\overline{X},\overline{Z})}{J_X(\overline{C},\overline{X},\overline{Z})}. \quad (35)$$

In evaluating $J_{CC}(\overline{C},\overline{X},\overline{Z})$ we use an additional policy simulation in which a 1 per cent shock is applied to C_t in the situation reached in the simulation that revealed $J_C(\overline{C},\overline{X},\overline{Z})$. The result for x_{t+1} in this additional simulation reveals $J_C(\overline{C}\times 1.01,\overline{X},\overline{Z})$. That is, the additional simulation shows the effect on next year's wealth of a 1 per cent increase in this year's consumption imposed in a situation in which consumption is 1 per cent above baseline and X_t and Z_t are on baseline. Now we can calculate $J_{CC}(\overline{C},\overline{X},\overline{Z})$ as

$$J_{CC}(\overline{C},\overline{X},\overline{Z}) = 100\frac{J_C(\overline{C}\times 1.01,\overline{X},\overline{Z}) - J_C(\overline{C},\overline{X},\overline{Z})}{J_C(\overline{C},\overline{X},\overline{Z})}. \quad (36)$$

In evaluating $J_{XX}(\overline{C},\overline{X},\overline{Z})$ we use an additional policy simulation in which a 1 per cent shock is applied to X_t in the situation reached in the simulation that revealed $J_X(\overline{C},\overline{X},\overline{Z})$. This enables us to calculate $J_{XX}(\overline{C},\overline{X},\overline{Z})$ as.

$$J_{XX}(\overline{C},\overline{X},\overline{Z}) = 100\frac{J_X(\overline{C},\overline{X}\times 1.01,\overline{Z}) - J_X(\overline{C},\overline{X},\overline{Z})}{J_X(\overline{C},\overline{X},\overline{Z})}. \quad (37)$$

Similarly, we conduct additional simulations to reveal off-baseline first-order

$$J_{CX}(\overline{C},\overline{X},\overline{Z}) = 100\frac{J_C(\overline{C},\overline{X}\times 1.01,\overline{Z}) - J_C(\overline{C},\overline{X},\overline{Z})}{J_C(\overline{C},\overline{X},\overline{Z})}, \tag{38}$$

$$J_{XZq}(\overline{C},\overline{X},\overline{Z}) = 100\frac{J_X(\overline{C},\overline{X},\overline{Z}_1,\overline{Z}_2,\ldots,\overline{Z}_{q-1},\ \overline{Z}_q\times 1.01,\overline{Z}_{q+1},\ldots,\overline{Z}_r) - J_X(\overline{C},\overline{X},\overline{Z})}{J_X(\overline{C},\overline{X},\overline{Z})}, \tag{39}$$

$q = 1,\ldots,r,$

$$J_{CZq}(\overline{C},\overline{X},\overline{Z}) = 100\frac{J_C(\overline{C},\overline{X},\overline{Z}_1,\overline{Z}_2,\ldots,\overline{Z}_{q-1},\ \overline{Z}_q\times 1.01,\overline{Z}_{q+1},\ldots,\overline{Z}_r) - J_C(\overline{C},\overline{X},\overline{Z})}{J_C(\overline{C},\overline{X},\overline{Z})}, \tag{40}$$

$q = 1,\ldots,r.$

elasticities that allow us to calculate the remaining second-order elasticities via equations (38) – (40).

V A DSGE Consumption Function for the USAGE Model of the USA

In this section we apply the theory from Sections II–IV to specify a DSGE consumption function for a 70-industry version of the USAGE model of the US economy. USAGE is a dynamic CGE model that was initially created at the Centre of Policy Studies in 2002. Since then, it has been applied and further developed by, and on behalf of: the US International Trade Commission; the US Departments of Commerce, Agriculture, Transportation, Homeland Security and Energy; the Canadian government; the Mitre Corporation; and the Cato Institute. Application topics include trade policies, illegal immigration, road/rail/air infrastructure, energy policies, and terrorism.[6]

In standard applications of USAGE, household consumption in year *t* is proportional to household disposable income in year *t* (fixed average propensity to consume, APC). Public consumption is usually linked in a linear way to private consumption. Investment in each industry in year *t* is a function of the industry's expected rate of return on capital. So that the model can be solved recursively, expected rates of return are assumed to reflect current rates of return.[7] Imports of each commodity are modelled as imperfect substitutes for domestically produced products in the same industrial classification – the Armington (1969) assumption. Exports of each commodity are modelled via constant-elasticity export demand functions.

As described in Section III, we equipped USAGE with a 3 per cent steady-growth baseline. The starting year for our baseline is 2018. Applying the perturbation method with this baseline, we estimated a DSGE consumption function that can be used in USAGE as an alternative to the standard fixed-APC function. Potentially the DSGE consumption function can be used to determine shock-induced deviations in private and public consumption from the 3 per cent steady-growth baseline caused by a wide variety of shocks (movements in the *Z* variables). Here we limit attention to just one *Z* variable: primary-factor-saving technology. Thus, our DSGE consumption function takes the form.

$$c_t = \mathrm{ELAST}(c,x) * x_t + \mathrm{ELAST}(c,z) * z_t \tag{41}$$

[6] There are many published USAGE application papers. Examples are Dixon *et al*. (2017a, 2017b).

[7] Dixon *et al*. (2005) show how models such as USAGE can be solved with forward-looking expectations for rates of return. The method involves a series of recursive dynamic simulations with adjustments in expectations between simulations.

In this equation:

- c_t is the percentage deviation in real consumption in year t from its baseline value. This is private plus public consumption deflated by a composite price index formed as a value weighted average of the prices indexes for private consumption and public consumption.
- x_t is the percentage deviation in real wealth at the start of year t from its baseline value. This is the deflated value of physical capital in the USA less net foreign liabilities. The deflator is the lagged value of the price index for private consumption; that is, the deviation in wealth at the start of year $t + 1$ is deflated by the deviation in the price of consumption in year t.
- z_t is the percentage deviation in primary-factor-saving technology in year t from its baseline value. This variable applies uniformly across all industries. If z_t equals 1, then all industries can produce their baseline level of output for year t with 1 per cent less primary-factor input than in the baseline and the baseline levels for intermediate inputs.
- ELAST(c, x) and ELAST(c, z), treated as parameters, are the elasticities of consumption (private plus public) with respect to start-of-year wealth and primary-factor-saving technology.

(i) Evaluation of the Coefficients in the DSGE Consumption Function for USAGE

We evaluate the two elasticities in Equation (41) according to the formulas in (32). These evaluations require us to assign values to:

- γ, the parameter introducing diminishing marginal utility to consumption and wealth in any year;
- β, the parameter introducing preference for current consumption relative to future consumption;
- J_X, J_C, J_Z, J_{XX}, J_{XC}, J_{XZ}, J_{CX}, J_{CC} and J_{CZ}, the first and second-order baseline elasticities of wealth at the start of year $t + 1$ with respect to wealth at the start of year t, consumption in year t and primary-factor-saving technology in year t;

Table 2
First-Order Real Wealth Elasticities

i	J_i
C	0.61227
X	0.98877
Z	0.71538

Table 3
Second-Order Real Wealth Elasticities

	$J_{i,s}$		
s	C	X	Z
i			
C	1.34142	−0.42973	−0.71970
X	0.27038	0.01713	−0.26245

- θ, the parameter introduced to allow utility in each year to be a function of wealth as well as consumption;
- M, the baseline value to lifetime welfare expected in year t from a unit increase in growth-discounted wealth at the start of year t;
- U_X, the baseline value to current utility from a unit increase in growth-discounted wealth at the start of year t; and
- M_X, the elasticity of the consuming agent's policy function (the M function) with respect to growth-discounted start-of-year wealth.

We set γ and β to 0.5 and 0.9. These values are representative of the values used in macro DSGE models.

In evaluating the J elasticities, we used USAGE simulations in which start-of-year real wealth, real consumption and primary-factor-saving technical change in 2018 are exogenous. We deduced the J elasticities, shown in Tables 2 and 3, by applying shocks to these three variables to determine their effects on real wealth at the start of 2019.

In these elasticity-evaluation simulations: shocks to wealth were imposed via increases above baseline in U.S. foreign assets at the start of 2018; shocks to consumption were imposed by equal percentage increases in 2018 above baseline in aggregate private and public consumption; and shocks to technology were imposed via uniform increases across industries in 2018 above baseline in primary-factor productivity.

Other exogenous variables in the simulations used to estimate the J elasticities included aggregate employment and aggregate capital. This is important for understanding the elasticity values in Tables 2 and 3 discussed later in this section. With aggregate employment and capital held constant, the only avenues for movements in real GDP are changes in technology and changes in dead-weight losses associated with taxes and other distortions such as differences in rates of return on capital across industries.

To evaluate θ, we use (13) and (20) with Xs, Cs and $\tilde{M}$ replaced by their growth-discounted baseline values which are constant through time. Omitting time subscripts to indicate baseline growth-discounted values, and recognising that on a non-stochastic steady-growth baseline $\tilde{M} = M$, we have.

$$\theta C^{\theta-1}X^{1-\theta}\left(X^{1-\theta}C^{\theta}\right)^{-\gamma} = \beta M J_C \frac{X}{C}. \quad (42)$$

and

$$M = (1-\theta)X^{-\theta}C^{\theta}\left(X^{1-\theta}C^{\theta}\right)^{-\gamma} + \beta\{M\}J_X \frac{X}{X}, \quad (43)$$

leading to

$$\theta = \frac{\beta J_C}{1-\beta(J_X-J_C)}. \quad (44)$$

With β assumed to be 0.9 and the values of J_C and J_X taken from Table 2, the value for θ obtained from (44) is 0.8335.

In the CGE database for the initial year (2018), the values for the X and C are \$US34.69 trillion and \$US16.58 trillion (see Table 4). M and U_X can now be evaluated using (43) and (3) via.

$$M = \frac{(1-\theta)X^{-\theta}C^{\theta}\left(X^{1-\theta}C^{\theta}\right)^{-\gamma}}{1-\beta J_X}, \quad (45)$$

TABLE 4
Key Items in the USAGE Database and Baseline

Concept	Value ($ trillion)
Baseline consumption (private plus public) in 2018	16.58
Baseline real wealth at the start of 2018	34.69
Baseline real wealth at the start of 2019	35.73
Baseline GDP in 2018	19.44

and

$$U_X = (1-\theta)X^{-\theta}C^{\theta}\left(X^{1-\theta}C^{\theta}\right)^{-\gamma}, \quad (46)$$

giving $M = 0.189$ and $U_X = 0.021$.

At this stage, the only parameter or coefficient in (32) to which we have not assigned a value is M_X. Recall from Section III that our strategy for evaluating M_X is to solve the seven equations in Table 1 with x_t set to 1 and z_t and $d\sigma$ set to 0; see (29). Determining M_X in this way uses only parameters and coefficients to which values have already been assigned. As explained in Appendix I, we obtain a quadratic equation for M_X. With the assigned parameter and coefficient values, the two solutions for this quadratic are:

$$M_X = -0.2388 \text{ and } M_X = -0.6362. \quad (47)$$

We know that M_X must be negative: diminishing marginal utility to wealth means that the increase in expected lifetime welfare from an additional unit of wealth must decline as wealth increases. However, that criterion does not help us chose between the two solutions in (47): both are negative. Consequently, we proceeded to (32) and evaluated ELAST(c, x) under each possible M_X value. With $M_X = -0.2388$ we obtained -0.5225, and with $M_X = -0.6362$ we obtained 0.2184. We require ELAST(c, x) to be positive: an increase in wealth in year t should generate an increase in consumption in year t. On this basis we chose the second solution in (47), $M_X = -0.6362$.

With M_X tied down we can now refer to (32) to evaluate ELAST(c, z). This leads to the following DSGE consumption function for USAGE:

$$c_t = 0.2184x_t + 0.6545z_t. \qquad (48)$$

(ii) Understanding the values for J *elasticities*

Using CGE simulations to estimate elasticities of the wealth-accumulation function (the J elasticities) is a new idea and involves non-standard closures of the CGE model. Consequently, it behoves us to examine the J elasticities, if for no other reason than to make sure they have been computed correctly.

Looking at the elasticities in Tables 2 and 3, and the data items in Table 4, our first question is why J_C is equal to 0.61227, rather than about 0.48 (=16.58/(35.73/1.03))? What explains the discrepancy of 0.13 between the actual value of J_C and what we would expect simply on the basis of consuming an amount of wealth worth 1 per cent of 2018 consumption?

The answer involves two factors. The first is that a 1 per cent increase in consumption in 2018 reallocates capital towards low-rate-of-return uses, especially housing. USAGE implies that this reduces GDP by 0.082 per cent, imparting a loss in next year's wealth of 0.04 per cent (= 0.082 × 19.44/(35.73/1.03)). The second factor is real appreciation: an increase in the price of non-traded goods relative to traded goods. This is necessary to facilitate the transfer of resources towards consumption away from the trade balance. (Recall that in the estimating simulation for J_C, there is little scope for increasing GDP.) The increase in the price of non-traded goods (e.g., housing services) relative to the price of traded goods generates an increase in the price of consumption relative to investment of about 0.12 per cent. The investment price index is the major price in determining the value of the capital stock (the main component of wealth), and the consumer price index for year t is the chosen deflator for determining the real value of wealth at the start of year t + 1. Thus, the movement in relative prices introduces a reduction in real wealth of about 0.12 per cent. Together these two factors suggest that the loss of real wealth at the start of 2019 should be about 0.16 per cent (= 0.04 + 0.12) greater than would be expected (0.48) on the basis of the relative sizes of consumption and wealth. This is close to the discrepancy of 0.13 per cent that we set out to explain.

Why is the value of J_X (= 0.989) less than 1? With the rate of interest and growth at 3 per cent, we expected a unit increase in wealth at the start of year t to translate into a unit increase in growth-discounted wealth at the start of t + 1, suggesting a value of 1. The reason J_X is a little less than 1 reflects the way in which we introduced the 1 per cent increase in wealth in the J_X estimating simulation. As mentioned earlier, we boosted wealth at the start of 2018 by 1 per cent via an increase in US foreign assets. Boosting wealth in this way reduced the average rate of return on US wealth. In the J_X estimating simulation this caused the simulated deviation from baseline in growth-discounted wealth at the start of 2019 to be slightly less than 1 per cent.

We anticipated that a 1 per cent primary-factor-saving technical change would increase GDP by about 1 per cent. So why is J_Z equal to 0.71, rather than 0.56 (=19.44/(35.73/1.03))? In the estimating simulation for J_Z, the trade balance moves towards surplus: there is an increase in real GDP and no increase in absorption. The movement towards trade surplus is facilitated by real devaluation, requiring a decrease in the price of non-traded goods relative to the price of traded goods. The movement in the non-traded/traded price ratio in the J_Z estimating simulation generates an increase in the price of investment relative the consumption of 0.14 per cent, and a corresponding increase in real wealth. This explains the discrepancy between 0.56 and 0.71.

Why is J_{CC} in Table 3 strongly positive, 1.34? J_C is the percentage damage to growth-discounted wealth at the start of 2019 from a 1 per cent increase in consumption in 2018. J_{CC} is the percentage difference between J_C evaluated with consumption in 2018 above its baseline value by 1 per cent and J_C evaluated on the baseline. If consumption is elevated 1 per cent above baseline, then a 1 per cent increase in consumption uses up about 1 per cent more of next year's wealth than if the consumption increase were just 1 per cent of baseline consumption. On this basis, we would expect J_{CC} to be about 1. However, if 2018 consumption is elevated 1 per cent above baseline, then 2019 growth discounted wealth will be below baseline by about 0.61227 per cent (the value of J_C). With 2019 growth-discounted wealth below baseline by 0.61227 per cent, any given destruction of wealth generated by consumption in 2018 produces a larger *percentage* effect on growth-discounted wealth than if wealth were on baseline. Now, we would expect J_{CC} to be about 1.62 (= 100 × 1.01/(1–0.0061227) – 1). But what takes J_{CC} from 1.62 to 1.34?

Elevating consumption in 2018 not only reduces wealth at the start of 2019, but also changes its composition. The elevation increases net foreign liabilities without changing domestic capital stocks. (Recall that in the elasticity-estimating simulations, aggregate capital is exogenous.) Thus, the elevation of consumption in 2018 increases the share of wealth in 2019 accounted for by physical capital (from 1.1819 in the baseline to 1.1881 in the simulation with elevated consumption in 2018). The increase in the domestic price level (real appreciation) accompanying a given increase in consumption is more beneficial to real wealth the higher the share of physical capital in wealth. This means that the inflation benefit to wealth is greater when we calculate $J_C\left(\overline{C}\times 1.01,\overline{X},\overline{Z}\right)$ than when we calculate $J_C\left(\overline{C},\overline{X},\overline{Z}\right)$. Taking this into account explains the lower than anticipated value for J_{CC}.[8]

J_{CX} is the percentage difference between J_C evaluated with real wealth at the start of year 2018 above its baseline value by 1 per cent and J_C evaluated on the baseline. The quantity of wealth depletion by the start of 2019 through a 1 per cent increase in consumption in 2018 does not depend on wealth. On this basis, we anticipated that with 1 per cent higher wealth at the start of 2018, the *percentage* damage to wealth at the start of 2019 would be reduced by about 1 per cent, that is, we anticipated that J_{CX} would be about −1. So why is J_{CX} greater than −1 (−0.43)?

Again, the answer is changes in the composition of wealth. This time, the share of wealth at the start of 2019 accounted for by physical capital is lowered by elevation of wealth at the start of 2018. Consequently, the inflation benefit to wealth is smaller when we calculate $J_C\left(\overline{C},\overline{X}\times 1.01,\overline{Z}\right)$ than when we calculate $J_C\left(\overline{C},\overline{X},\overline{Z}\right)$, explaining the higher than anticipated value for J_{CX}.

Continuing in this way, we could explain all of the items in Table 3. However, we have done enough to be convinced that the computations underlying Table 3 are correct. The most important point about the explanations is that simple intuition is confounded by changes in relative prices and by seemingly innocuous but arbitrary assumptions concerning the composition of changes in wealth.

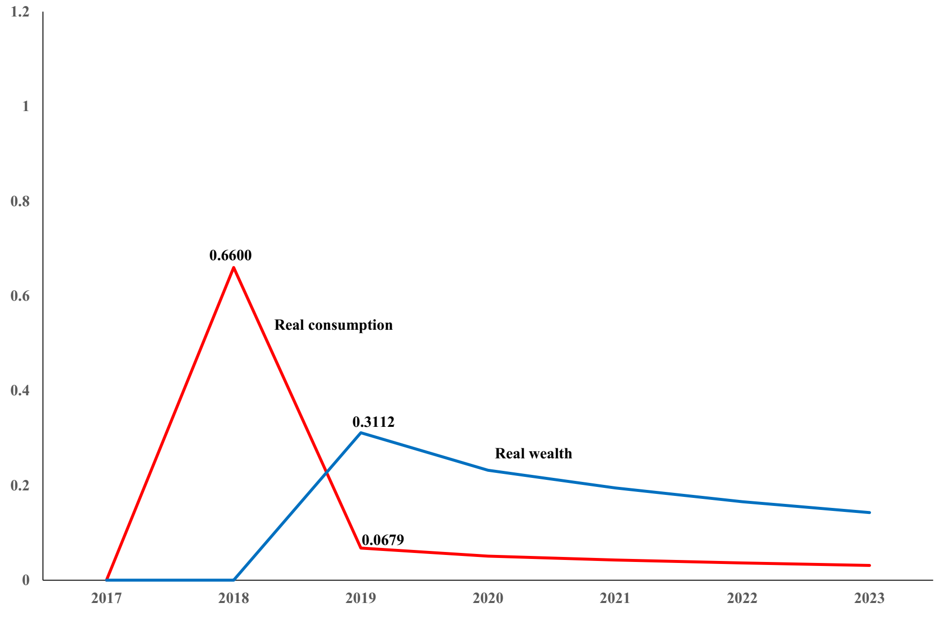
VI Illustrative Application: The Effects of a 1 Per cent Shock to Primary-Factor-Saving Technology

Figure 1 shows USAGE results for the effects of a 1 per cent deviation in primary-factor-saving technology from its baseline path occurring in 2018. The shock is temporary: primary-factor-saving technology is expected to return to its baseline path from 2019 onwards.

The closure for this simulation is different from that used in the estimation of the J elasticities. Now consumption is endogenous, determined by Equation (48). Aggregate employment remains exogenous (unaffected by the shock) and capital is predetermined (and therefore also unaffected by the shock in the first year). With aggregate employment and capital fixed, a 1 per cent improvement in primary-factor-saving technology must cause a deviation in GDP in 2018 from its baseline path of approximately 1 per cent. In

[8] Our simulations show that elevating consumption in 2018 by 1 per cent increases the price index for investment by 0.256 per cent. This imparts a percentage increase in wealth in the $J_C\left(\overline{C}\times 1.01,\overline{X},\overline{Z}\right)$ simulation relative to the $J_C\left(\overline{C},\overline{X},\overline{Z}\right)$ simulation of 0.00161 [=(1.1882–1.1819) × 0.256]. This adjustment reduces our back-of-the-envelope estimate of J_{CC} by 0.26 [=100 × 0.00161/J_C = 100 × 0.00161/0.61], bringing it closely into line with the value in Table 3.

FIGURE 1
Effects of a 1 per cent Temporary Improvement in Primary-Factor Technology with a DSGE Consumption Function (Percentage Deviations from Baseline)



fact, the USAGE result was an increase of 1.03 per cent.[9]

An increase in GDP in 2018 of 1.03 per cent is worth \$0.20 trillion (= 19.44 × 0.0103). The increase in consumption dictated by Equation (48) is 0.6600 per cent.[10] This uses up \$0.11 trillion of the GDP increase (=16.58 × 0.006600), leaving \$0.09 trillion as a contribution to an increase in wealth at the start of 2019. This contribution is a percentage increase in real wealth of 0.25 per cent (=100 × 0.09/35.73). The actual increase projected by USAGE and shown in Figure 1 is 0.31 per cent. The extra 0.06 per cent (=0.31 – 0.25) comes from price changes. As explained in Section V in our discussion of the value of J_Z, a primary-factor saving improvement in technology generates an increase in the price of capital goods relative to consumption goods. With our chosen price deflator for real wealth at the start of 2019 being the price deflator for consumption in 2018 and the price of wealth being predominately the price of capital goods, the relative price movement in 2018 imparts an increase in real wealth at the start of 2019. In the USAGE simulation, the increase in the price of capital goods relative to the price of

[9] The discrepancy of 0.03 percentage points is not important for illustrating the workings of our DSGE consumption function. Nevertheless, we traced the source to a reallocation of the capital stock in 2018 towards industries that happen to have relatively high rates of return on capital.

[10] At first glance, Equation (48) appears to dictate an increase of 0.6545 per cent. However, in the GEMPACK software which we use, (48) is interpreted as the nonlinear equation $\frac{C}{Cbase} = \left(\frac{RWEALTH}{RWEALTHbase}\right)^{0.2184}\left(\frac{TECH}{TECHbase}\right)^{-0.6545}$, where TECH is primary factor input per unit of output. In 2018, the first term in parentheses on the right-hand is 1 and the second term in parentheses is 0.99. This produces a percentage consumption deviation of 0.6600 [$=100 \times (0.99^{-0.6545} - 1)$].

consumption goods in 2018 is 0.07 per cent,[11] closely explaining the bonus real wealth increase (0.06 per cent) beyond that generated by extra saving in 2018.

As mentioned in Section I, in standard applications of USAGE private and public consumption move proportionately with income. Under this treatment, the benefits of a good-news temporary shock are absorbed almost entirely as an immediate increase in consumption. In our DSGE specification, the year t contribution to lifetime welfare is a diminishing-marginal-utility function of wealth at the start of year t and consumption in year t; see Equation 3. With diminishing marginal utility, we anticipated that the replacement in USAGE of the standard consumption function by a DSGE consumption function would spread the consumption response to temporary good-news across years. With time-preference discounting (β in (2) is 0.9) we anticipated that the DSGE deviation path for consumption would be declining after the first year, but in a relatively smooth manner.

Figure 2 compares the DSGE results from Figure 1 with results under a standard USAGE consumption function. With the standard USAGE treatment, the consumption deviation is 0.9589 per cent in 2018, falling to 0.0210 per cent in 2019. With the DSGE treatment, the consumption deviation is 0.6600 per cent in 2018, falling to 0.0679 per cent in 2019. Thus, as anticipated, the introduction of the DSGE consumption function has a smoothing effect on the consumption deviations and, as was also anticipated, the consumption deviations decline over time.

Although the introduction of the DSGE consumption function smooths out the consumption response to the temporary shock to primary-factor technology, we were surprised that the smoothing was not more pronounced. Even with the DSGE consumption specification, the 2018 consumption deviation is 9.72 times the 2019 deviation (0.6600/0.0679).

Why, even with the DSGE specification, does consumption increase so sharply in 2018 relative to 2019? There are three reasons.

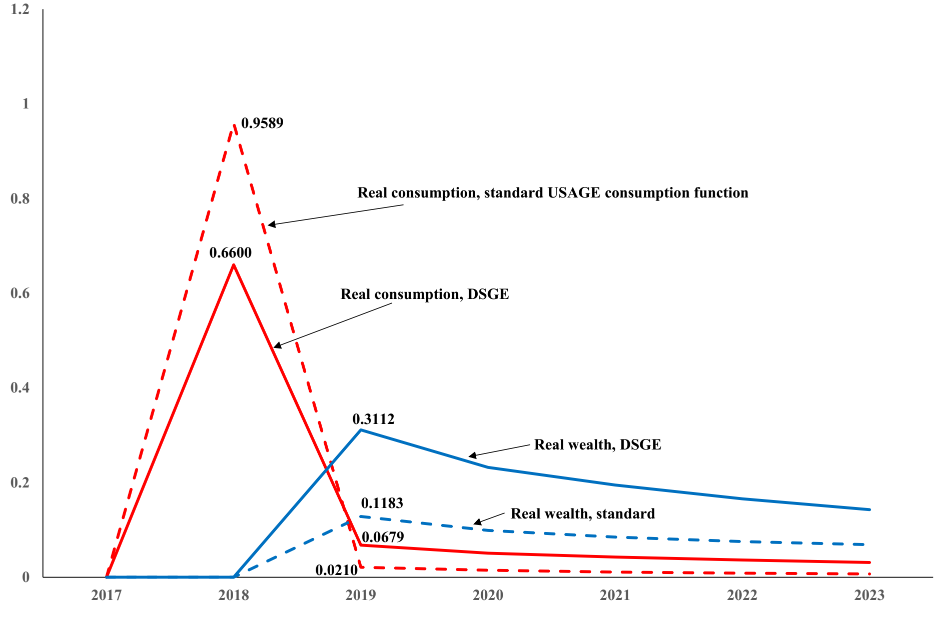
The first is that all of the benefit to be enjoyed in 2018 from the temporary shock must be generated by a consumption increase. Wealth in 2018 is predetermined. From 2019 onwards, some of the benefit can be taken in the form of extra wealth. Thus, to smooth out utility contributions through time, the DSGE consuming agent must make a relatively large consumption increase in 2018 when this is the only avenue for generating utility. However, this is not the whole story. As shown in Figure 3, the deviation path for the annual utility contribution is far from smooth. The utility deviation in 2018 is 5.06 times that in 2019 (=0.2745/0.0542).

The second reason for the large consumption deviation in 2018 relative to that in 2019 relates to relative prices in 2018 compared with 2019 and later years. As explained already, the primary-factor technology shock in 2018 generates a reduction in the price of consumption goods relative to the price of capital goods. This effect on relative prices is temporary, creating an incentive for increased consumption in 2018 when consumption goods are relatively cheap.

The third reason relates to the increase in real wealth at the start of 2019 generated by the relative price change in 2018. This is similar to a gift at the start of 2019 that is withdrawn in subsequent years. There is strongly diminishing marginal utility to extra wealth in any given year ($(1-\theta)$ in Equation 3 is 0.1665). The gift of wealth at the start of 2019 makes it difficult to transfer utility from 2018 to 2019 through extra saving in 2018. Thus, the consuming agent takes a disproportionate share of the good news from the temporary technology improvement in 2018 as a utility increase in 2018, rather than as a utility increase in subsequent years.

[11] The increase in the price of capital goods relative to consumption goods in the simulation that revealed J_Z was 0.14 per cent. In the simulation being discussed here, it is only 0.07 per cent. In generating J_Z we held consumption constant. In the current simulation, consumption moves. This dampens the increase in the ratio of the price of capital goods to the price of consumption goods.

FIGURE 2
Effects of a 1 per cent Temporary Improvement in Primary-Factor Technology with DSGE and Standard USAGE Consumption Functions (Percentage Deviations from Baseline)



VII Key Sensitivities

Readers of earlier drafts of this paper have wondered about two aspects of the application in Section VI: the effect of including wealth as an argument of the utility function; and the lack of persistence in the shock. In this section, we generate results comparable with those in Section VI, but with a reduced weight for wealth in utility and with persistence in the primary factor technology shock.

To reduce the importance of wealth in the utility function, we reset β (the time-preference discount rate) to 0.95, up from 0.90 in the application in Section VI. Via Equation (44) this leads to a recalibrated value for $1-\theta$ of 0.0944, down from 0.1665. In 3, this takes the utility function a considerable part of the way to the standard case in which wealth is excluded. We cannot exclude wealth entirely by allowing $1-\theta$ to be 0. This is because with J_X less than 1 (see Table 2),[12] a value of 1 for θ in (44) would require a value for β that is greater than 1, defying the logic of time-preference discounting.

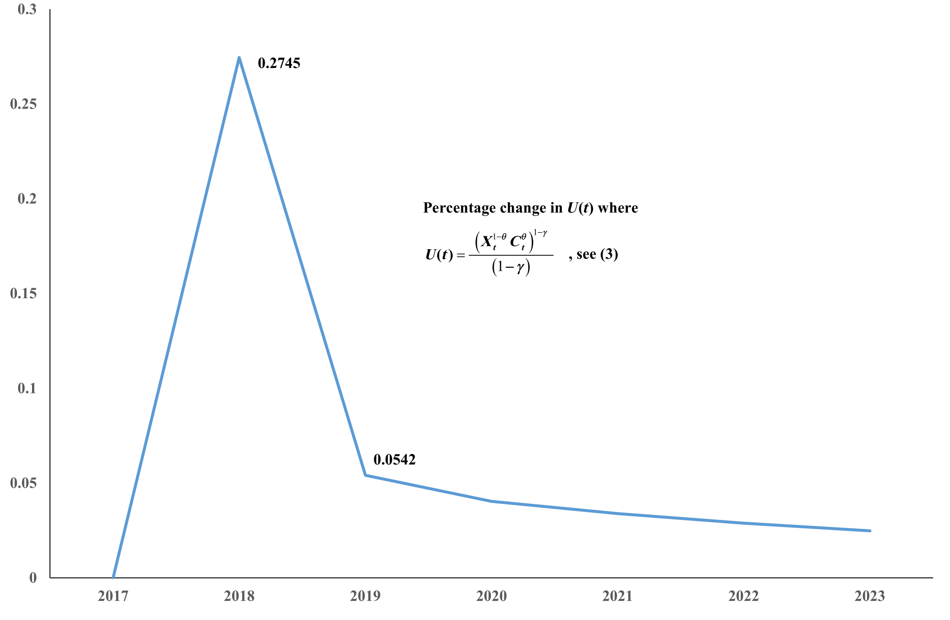
To explore the implications of shock persistence, we specify the evolution of the Z vector by.

$$Z_{t+1} = \rho Z_t + \sigma \varepsilon_{t+1}, \quad (49)$$

where ρ is a non-negative parameter less than 1; ε_{t+1} is a vector of uncorrelated draws from a distribution with mean 0 and variance 1; and σ is a variance–covariance matrix. In the application in Section VI, ρ was implicitly set to 0.

[12] Note that the value of J_X does not depend on the value of θ or β.

FIGURE 3
Effects on Annual Utility of a 1 per cent Temporary Improvement in Primary-Factor Technology with a DSGE Consumption Function (Percentage Deviations from Baseline)



With the introduction of a non-zero value for ρ, there is just one revision to the equations in Table 1: Equation (27) has an additional terms and becomes.

$$\tilde{m}(t) = M_X x_{t+1} + M_\sigma d\sigma + \rho M_Z z_t. \qquad (50)$$

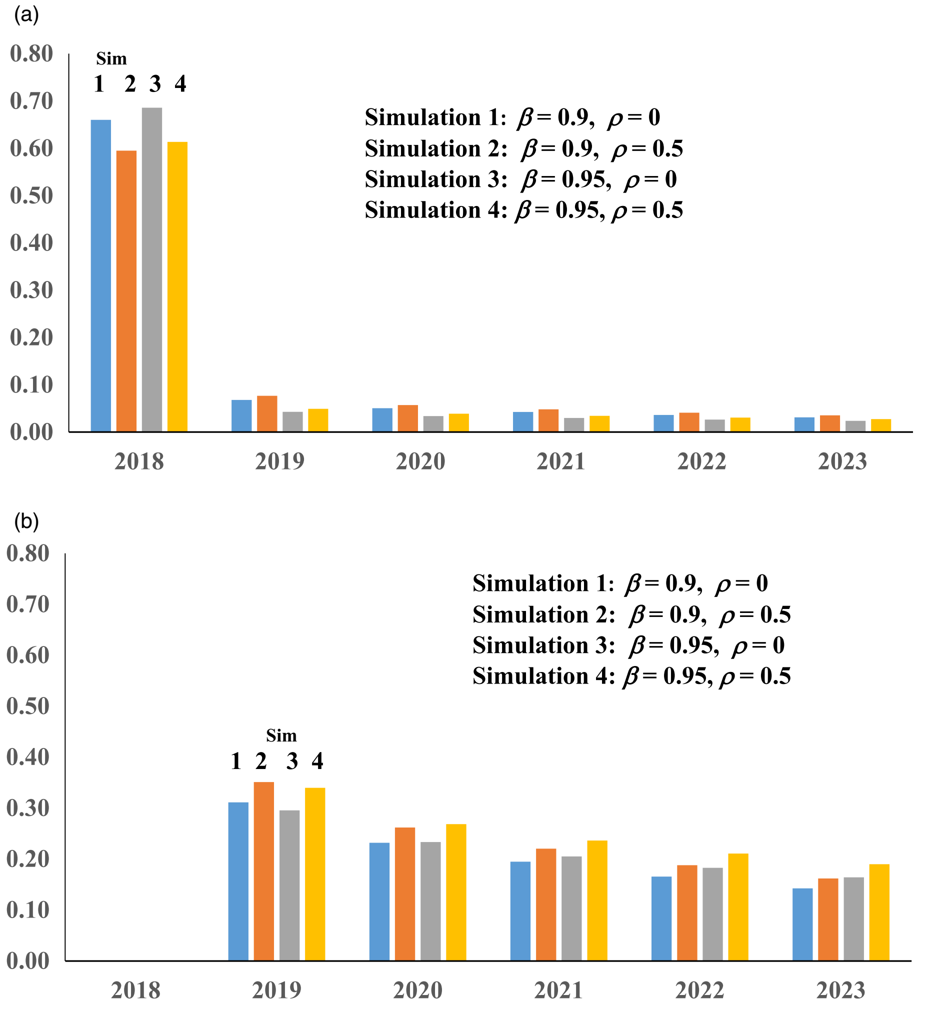
The additional term does not affect the wealth coefficient (ELAST(c, x)) in the consumption function (41), but it does affect the coefficient on the shock variable (ELAST (c, z)). As shown in Appendix I, ELAST(c, x) depends on β (via θ) but is independent of ρ, and ELAST(c, z) depends on both β and ρ.

Table 5 shows ELAST values for four combinations of values of β (and hence θ) and ρ. The values in the first row are the settings used in the application reported in Section VI. We used the values in rows 2–4 in three sensitivity simulations. Figure 4 gives results for real consumption and real wealth from the simulation reported in Section VI, together with results from the three sensitivity simulations.

Figure 4 shows that the main features of the results from Section VI are retained in the sensitivity simulations. The smoothing effect of DSGE behaviour in our simulations remains weak. As reported in Section VI for the central simulation (Sim1 in Table 5), the consumption deviation in 2018 is 9.72 times the deviation in 2019. In the three sensitivity simulations, the 2018 deviations are 7.76, 15.97 and 12.44 times the corresponding 2019 deviations.

Comparing simulations 1 and 3 or simulations 2 and 4 shows the effects of reducing the importance of wealth in the utility function. Reducing $1-\theta$ from 0.1665 to 0.0944 by increasing β from 0.9 to 0.95 increases the 2018 consumption deviations, but only slightly. On the one hand, reducing $1-\theta$ stimulates current consumption from a good-news shock by reducing the welfare

Figure 4
Effects on (a) Consumption, and (b) Real Wealth, of a 1 per cent Improvement in Primary-Factor Technology in 2018 with Different Assumptions for Time Discounting (β) and Shock Persistence (ρ) (Percentage Deviations from Baseline)



value of holding wealth. On the other hand, increasing β reduces current consumption by increasing the welfare value of future consumption. In our sensitivity simulations, these two effects approximately balance in the determination of the immediate consumption effect of a productivity increase in 2018. In interpreting this result, it is useful to recall that $1-\theta$ and β are related by Equation (44) and that the determination of the J elasticities appearing in (44) depends only on data and our assumption of 3 per cent baseline growth, not on $1-\theta$ and β. Calibrating our model through different combinations of $1-\theta$ and β compatible with (44) affects ELAST(c, z); see Table 5. But the effect is small because whatever combination of $1-\theta$ and β is chosen, the specification of consumer preferences must be consistent with the 2018 data on wealth and consumption, and with 3 per cent baseline growth.

With more consumption in 2018 as we go from simulation 1 to simulation 3 (or 2 to 4), there is less wealth at the start of 2019. Beyond 2019, wealth is a little higher in simulation 3 than in simulation 1 (and in 4 than in 2). This is explained by lower consumption from 2019 onwards in simulation 3 relative to 1 (and in 4 relative to 2).

Comparing simulations 1 and 2 or simulations 3 and 4 shows the effects of shock persistence. We expected increased likelihood of good news in 2019 (continued likely elevated productivity) to generate increased consumption from the good news in 2018. However, increasing ρ from 0 to 0.5 decreases the 2018 consumption deviations. The explanation relates to the real devaluation required to move the trade balance to surplus in light of only partial consumption in 2018 of the productivity-related income gain. As we pointed out in Section V in the discussion of J_Z and again in Section VI, real devaluation associated with a good-news productivity shock in 2018 generates a decrease in the price of consumption relative to the price of investment. With the expectation that the good news will not persist ($\rho = 0$) in 2019, the price movements in 2018 provide a strong incentive to front-load consumption. When we assume that part of the good news is likely to persist ($\rho = 0.5$), implying that part of the price movements from 2018 are likely to persist into 2019, then the incentive to front-load consumption is diminished.

In simulation 2, reduced consumption in 2018 and persisting higher productivity in the years beyond 2018 generate higher real wealth in 2019 and in all future years relative to simulation 1. This also applies to simulation 4 relative to 3. Increased wealth and productivity support higher consumption levels in 2019 and future years in simulation 2 relative to simulation 1 and in simulation 4 relative to simulation 3.

Finally, we compare simulations 1 and 4. In going from simulation 1 to simulation 4, the consumption deviation in 2018 is reduced: the consumption increase from the change in β and the consequent change in $1-\theta$ is outweighed by the consumption decrease from the change in ρ. Consumption in simulation 4 remains slightly below that in simulation 1 for the other years in Figure 4a. Correspondingly, wealth in all years is higher in simulation 4 than in simulation 1.

VIII Concluding Remarks

In DSGE modelling, agents make decisions in year t by applying rules (policy functions) that take account of: year t values of predetermined stock variables; year t values of exogenous variables; accumulation relationships determining future values of stock variables; and probability distributions for future values of exogenous variables. The key idea in DSGE modelling is that under rational expectations agents know that the rules they apply in year t will also be applicable in future years.

DSGE models generally have little sectoral, trade, technology and tax detail. Nevertheless, we find DSGE ideas attractive. This has led to the research reported in this paper in

Table 5

Parameter Values for Four Simulations with DSGE Consumption Functions

Sim no.	β, time discount Equation (2)	$(1-\theta)$, wealth utility Equations (3) and (44)	ρ, Persistence Equation (49)	ELAST(c, x) Equation (41)	ELAST(c, z) Equation (41)
1	0.9	0.1665	0	0.2184	0.6545
2	0.9	0.1665	0.5	0.2184	0.5902
3	0.95	0.0944	0	0.1452	0.6797
4	0.95	0.0944	0.5	0.1452	0.6085

which we incorporate a DSGE consumption function in a CGE model that contains a high level of sectoral disaggregation and considerable detail on trade, technology and taxes.

Making DSGE ideas operational requires numerical determination of policy functions that describe agent behaviour in year *t*. For CGE modelling, especially with GEMPACK software, the perturbation method seems the most natural way to determine these policy functions.

The perturbation method for finding the policy rule requires evaluation of either derivatives or elasticities of the wealth-accumulation relationship. In small models, the required elasticities can often be evaluated via formulas expressed in terms of known parameters. However, this option is not available for a full-scale CGE model. To overcome this problem we showed how the elasticities can be evaluated by suitable CGE simulations. For example, to obtain the elasticity of start-of-year wealth for year $t+1$ with respect to primary-factor-saving technology, we conducted a simulation in which primary-factor-saving technology in year *t* was shocked by 1 per cent and other exogenous variables and consumption were held fixed. The deviation result for start-of-year wealth in year $t+1$ revealed the required elasticity.

With elasticities of year $t+1$wealth evaluated by CGE simulations, we were able to compute coefficients for linearised (first-order approximation) DSGE consumption functions. We used these in illustrative CGE simulations of the effects of improvements in primary-factor-saving technology. The technology shocks produced an increase in income in year *t*. Under the usual CGE specification in which consumption moves in line with income, an income gain in year *t* is almost entirely consumed in year *t*. By contrast, with DSGE consumption functions, some of the income gain is devoted to wealth accumulation, allowing consumption benefits to be spread across time.

While the spread effect was clearly visible in our CGE simulations with DSGE consumption, it was weak. The introduction of a DSGE consumption function did not prevent a high proportion of the income gain in year *t* from being consumed in year *t*. The main explanatory factor was a temporary increase in the price of capital goods relative to consumption goods. This is a CGE effect that would not be captured by a small-scale DSGE model.

Our illustrative CGE simulations with a DSGE consumption function demonstrate the feasibility of transferring key DSGE ideas into a full-scale CGE model. We think that the integration of these two types of models has the potential to produce insights of value to researchers in both modelling streams. Our illustrative simulations raise the possibility for CGE modellers of adopting consumption functions that imply intertemporal spreading of consumption effects tailored to specific shocks. For DSGE modellers, they show the potential importance of relative price effects.

There are many directions in which the research in this paper could be extended. For us, the most interesting possibilities concern applications of a DSGE-modified CGE model to typical CGE questions on trade, environment, tax, technology and labour markets. For more technically attuned DSGE specialists, the challenge of assessing the accuracy or otherwise of our first-order-approximate consumption functions might be appealing.

Conflict of Interest

No conflict of interest.

REFERENCES

Aguiar, A., Corong, E. and van der Mensbrugghe, D. (2019), *The GTAP Recursive Dynamic (GTAP-RD) Model: Version 1.0*. Center for Global Trade Analysis, Purdue University, West Lafayette, IN. Available from: https://www.gtap.agecon.purdue.edu/resources/download/9871.pdf.

Armington, P. (1969), 'A Theory of Demand for Products Distinguished by Place of Production', *Staff Papers-International Monetary Fund*, **16** (1), 159–178.

Bouakez, H., Cardia, E. and Ruge-Murcia, F.J. (2009), 'The Transmission of Monetary Policy in a Multisector Economy', *International Economic Review*, **50** (4), 1243–66.

Corong, E., Hertel, T., McDougall, R., Tsigas, M. and van der Mensbrugghe, D. (2017), 'The Standard GTAP Model, Version 7', *Journal of Global Economic Analysis*, **2**, 1–119.

Dixon, P.B., Koopman, R.B. and Rimmer, M.T. (2013), 'The MONASH Style of CGE Modeling: A Framework for Practical Policy Analysis', Chapter 2, in Dixon, P.B. and

Jorgenson, D.W. (eds), *Handbook of Computable General Equilibrium Modeling*. Elsevier, Waltham, MA; 23–102.
Dixon, P.B., Pearson, K.R., Picton, M.R. and Rimmer, M.T. (2005), 'Rational Expectations for Large CGE Models: A Practical Algorithm and a Policy Application', *Economic Modelling*, **22** (6), 1001–19.
Dixon, P.B. and Rimmer, M.T. (2002), *Dynamic General Equilibrium Modelling for Forecasting and Policy: A Practical Guide and Documentation of MONASH, Contributions to Economic Analysis 256*. North-Holland Publishing Company, Amsterdam, The Netherlands, Xiv+338.
Dixon, P.B. and Rimmer, M.T. (2020), ' Developing a DSGE Consumption Function for a CGE Model', CoPS/IMPACT Working Paper G-296, 53. Available from: https://www.copsmodels.com/ftp/workpapr/g-296.pdf.
Dixon, P.B., Rimmer, M.T. and Waschik, R. (2017a), 'Evaluating the Effects of Local Content Measures in a CGE Model: Eliminating the US Buy America(n) Programs', *Economic Modelling*, **68**, 155–166.
Dixon, P.B., Rimmer, M.T. and Waschik, R. (2017b), 'Linking CGE and Specialist Models: Deriving the Implications of Highway Policy Using USAGE-Hwy', *Economic Modelling*, **66**, 1–18.
Hertel, T.W. (ed) (1997), *Global Trade Analysis: Modeling and Applications*. Cambridge University Press, Cambridge, UK, xvii +403.
Hornstein, A. and Praschnik, J. (1997), ' Intermediate Inputs and Sectoral Comovement in the Business Cycle', *Journal of Monetary Economics*, **40** (3), 573–595.
Horridge, J.M., Jerie, M., Mustakinov, D. and Schiffmann, F. (2018), GEMPACK Manual, ISBN 978-1-921654-34-3, Available from: https://www.copsmodels.com/gpmanual.htm.
Horridge, M., Meeraus, A., Pearson, K.R. and Rutherford, T. (2013), 'Software Platforms: GAMS and GEMPACK', Chapter 20, in Dixon, P.B. and Jorgenson, D.W. (eds), *Handbook of Computable General Equilibrium Modeling*. Elsevier, Amsterdam; 1331–82. https://doi.org/10.1016/B978-0-444-59568-3.00020-1
Rees, D.M., Smith, P. and Hall, J. (2016), 'A Multi-Sector Model of the Australian Economy', *Economic Record*, **92** (298), 374–408.
Schmitt-Grohé, S. and Uribe, M. (2004), 'Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function', *Journal of Economic Dynamics and Control*, **28** (4), 755–775.
Villaverde, J., Rubio Ramirez, J. and Schorfheide, F. (2016), 'Solution and Estimation Methods for DSGE Models', Chapter 9, in Taylor, J.B. and Uhlig, H. (eds), *Handbook of Macroeconomics*, vol. **2A**. Elsevier, Amsterdam, The Netherlands; 527–724.

Appendix I
Deriving the M *Elasticities and the Consumption Function*

We determine the values of M_X, M_σ and M_Z by using the equations in Table 1 but with Equation (27) replaced by Equation (50). Thus, we allow for simulations in which $\rho \neq 0$.

We also derive formulas for the ELAST coefficients in (41).

(i) Finding the value of $\mathrm{M_X}$

As explained in Section III, we derive M_X by applying (29) in Table 1. Under (29), the revision to (27) has no effect: with $z_t = 0$, the additional term in (50) is zero. This is sufficient to demonstrate that M_X is independent of ρ.

Under condition (29), we obtain a quadratic expression for M_X by eliminating the other six unknowns from the seven equations in Table 1. To do this, we apply (29) and start by using (26), (50), (16) and (22) to eliminate $m(t)$, $\tilde{m}(t)$, $j_C(t)$ and $j_X(t)$ from (15) and (21):

$$(\theta-1-\theta\gamma)c_t + (1-\theta)(1-\gamma) = M_X x_{t+1} + J_{CX} + J_{CC}c_t + x_{t+1} - c_t, \quad \text{(A1)}$$

$$M^* M_X = U_X[(\theta\gamma-\theta-\gamma) + (1-\gamma)\theta c_t] + \beta M J_X\left[M_X x_{t+1} + J_{XX} + J_{XC}c_t + x_{t+1} - 1\right]. \quad \text{(A2)}$$

Rearranging (9), we obtain

$$c_t = \left(\frac{x_{t+1} - J_X}{-J_C}\right). \quad \text{(A3)}$$

Substituting from (A3) into (A1) and (A2) gives

$$\frac{\{-J_X(\theta-\theta\gamma-J_{CC})-J_C[(1-\theta)(1-\gamma)-J_{CX}]\}}{\{-J_C[M_X+1]-(\theta-\theta\gamma-J_{CC})\}}=x_{t+1} \tag{A4}$$

and

$$\begin{aligned}&\{MM_X-\beta MJ_X(J_{XX}-1)-U_X(\theta\gamma-\theta-\gamma)\}\\&=[U_X(1-\gamma)\theta+\beta MJ_XJ_{XC}]\left(\frac{x_{t+1}-J_X}{-J_C}\right)\\&\quad+[\beta MJ_X(M_X+1)]x_{t+1}.\end{aligned} \tag{A5}$$

Rearrange (A5) to obtain

$$\frac{\begin{bmatrix}J_CMM_X-J_C\beta MJ_X(J_{XX}-1)-J_CU_X(\theta\gamma-\theta-\gamma)\\-[J_XU_X(1-\gamma)\theta+J_X\beta MJ_XJ_{XC}]\end{bmatrix}}{\{[J_C\beta MJ_X(M_X+1)]-[U_X(1-\gamma)\theta+\beta MJ_XJ_{XC}]\}}=x_{t+1}. \tag{A6}$$

Combine (A4) and (A6) to eliminate x_{t+1}:

$$\begin{aligned}&\frac{\begin{bmatrix}J_CMM_X-J_C\beta MJ_X(J_{XX}-1)-J_CU_X(\theta\gamma-\theta-\gamma)\\-[J_XU_X(1-\gamma)\theta+J_X\beta MJ_XJ_{XC}]\end{bmatrix}}{\{[J_C\beta MJ_X(M_X+1)]-[U_X(1-\gamma)\theta+\beta MJ_XJ_{XC}]\}}\\&\quad=\frac{\{-J_X(\theta-\theta\gamma-J_{CC})-J_C[(1-\theta)(1-\gamma)-J_{CX}]\}}{\{-J_C[M_X+1]-(\theta-\theta\gamma-J_{CC})\}}.\end{aligned} \tag{A7}$$

Cross-multiply in (A7):

$$\begin{aligned}&\begin{bmatrix}J_CMM_X-J_C\beta MJ_X(J_{XX}-1)-J_C{}^*[U_X(\theta\gamma-\theta-\gamma)]\\-[J_XU_X(1-\gamma)\theta+J_X\beta MJ_XJ_{XC}]\end{bmatrix}\Big\{-J_CM_X-J_C-(\theta-\theta\gamma-J_{CC})\Big\}\\&=\Big\{-J_X(\theta-\theta\gamma-J_{CC})-J_C[(1-\theta)(1-\gamma)-J_{CX}]\Big\}\\&\quad\times\Big\{J_C\beta MJ_XM_X+J_C\beta MJ_X-[U_X(1-\gamma)\theta+\beta MJ_XJ_{XC}]\Big\}.\end{aligned} \tag{A8}$$

Working with (A8), we find that M_X satisfies the quadratic equation

$$e_2M_X^2+e_1M_X+e_0=0, \tag{A9}$$

where

$$e_2=\{J_C^2M\}, \tag{A10}$$

$$\begin{aligned}e_1=&\{-J_X(\theta-\theta\gamma-J_{CC})-J_C[(1-\theta)(1-\gamma)-J_{CX}]\}\{J_C\beta MJ_X\}\\&+\left\{\begin{matrix}-J_C\beta MJ_X(J_{XX}-1)-[U_X(1-\gamma)\theta+\beta MJ_XJ_{XC}]J_X\\-J_CU_X[(\theta\gamma-\theta-\gamma)]\end{matrix}\right\}\{J_C\}\\&+\{J_CM\}\{J_C+(\theta-\theta\gamma-J_{CC})\},\end{aligned} \tag{A11}$$

$$e_0 = \{-J_X(\theta-\theta\gamma-J_{CC})-J_C[(1-\theta)(1-\gamma)-J_{CX}]\}\Big\{J_C\beta MJ_X - [U_X(1-\gamma)\theta+\beta MJ_XJ_{XC}]\Big\} + \left\{\begin{matrix} -J_C\beta MJ_X(J_{XX}-1)-[U_X(1-\gamma)\theta+\beta MJ_XJ_{XC}]J_X \\ -J_CU_X[(\theta\gamma-\theta-\gamma)] \end{matrix}\right\}\{J_C+(\theta-\theta\gamma-J_{CC})\}. \quad \text{(A12)}$$

Equations (A9)–(A12) will normally give two real solutions for M_X. Which should we choose? If the solutions differ in sign, we choose the negative solution: an increase in X_t reduces the value of an extra unit of wealth. In the application reported in Section V (see Equation (47)), both solutions were negative. As explained there, we chose the solution that led to a positive elasticity for consumption with respect to wealth. We were able to reject the other solution because it led to a negative elasticity for consumption with respect to wealth.

(ii) Finding the Value of M_σ

Under condition (30), we obtain an equation for M_σ of the form

$$\text{Coefficient} \times M_\sigma = 0. \quad \text{(A13)}$$

In general, Coefficient will not be zero, leading us to the conclusion that M_σ must be zero. To derive (A13) we apply (30) and start by substituting from (9), (16), (22), (26) and (50) into (15) and (21) to eliminate x_{t+1}, $j_C(t)$, $j_X(t)$, $m(t)$ and $\tilde{m}(t)$. This gives

$$\{(\theta-\theta\gamma)+(M_X+1)J_C-J_{CC}\}c_t = M_\sigma \quad \text{(A14)}$$

and

$$(1-\beta J_X)M_\sigma = \left\{\frac{U_X}{M}[(1-\gamma)\theta]+\beta J_X[-(M_X+1)J_C+J_{XC}]\right\}c_t. \quad \text{(A15)}$$

Eliminating c_t leads to an equation of the form (A13):

$$\left[\frac{\{(\theta-\theta\gamma)+(M_X+1)J_C-J_{CC}\}(1-\beta J_X)}{\left\{\frac{U_X}{M}[(1-\gamma)\theta]+\beta J_X[-(M_X+1)J_C+J_{XC}]\right\}} - 1\right]M_\sigma = 0. \quad \text{(A16)}$$

Equation (A16) means that either the coefficient on the left is zero or M_σ is zero. The coefficient is formed using data items and parameters. In general, it will not be zero. This can be checked by evaluating it in any application. Hence, we conclude that $M_\sigma = 0$.

(iii) Finding the Value of M_Z

Having derived a formula for M_X and having shown that $M_\sigma = 0$, we derive a formula for M_Z. This formula allows M_Z to be evaluated after the evaluation of M_X. In obtaining the formula for M_Z we set.

$$x_t = 0, z_t = 1 \text{ and } d\sigma = 0. \quad \text{(A17)}$$

Applying (A17), we start by using (16), (22), (26), (50) and (9) to eliminate $j_c(t)$, $j_x(t)$, $m(t)$, $\tilde{m}(t)$ and $x_{t+1.}$ This leads to

$$(\theta-1-\theta\gamma)c_t = \{M_X+1\}[-J_Cc_t+J_Z] + \rho M_Z + [J_{CC}-1]c_t + J_{CZ} \quad \text{(A18)}$$

and

$$M_Z = \frac{U_X}{M}(1-\gamma)\theta c_t + \beta J_X(M_X+1)[-J_Cc_t+J_Z] + \beta J_X\rho M_Z + \beta J_XJ_{XC}c_t + \beta J_XJ_{XZ}. \quad \text{(A19)}$$

Recalling that M_X is already known, we see that (A18) and (A19) provide two equations in two unknowns, M_Z and c_t. From (A18) we obtain.

$$c_t = \frac{\rho}{[(\theta-\theta\gamma)+(M_x+1)J_c-J_{cc}]}M_z + \frac{(M_x+1)J_z+J_{cz}}{[(\theta-\theta\gamma)+(M_x+1)J_c-J_{cc}]} \quad \text{(A20)}$$

Then we substitute from (A20) into (A19) to eliminate c_t:

$$\left\{(1-\beta J_X\rho)+\frac{\left[-\frac{U_X}{M}(1-\gamma)\theta+\beta J_X(M_X+1)J_C-\beta J_X J_{XC}\right]}{[(\theta-\theta\gamma)+(M_X+1)J_C-J_{CC}]}\rho\right\}M_Z$$
$$=\beta J_X(M_X+1)J_Z+\beta J_X J_{XZ} - \frac{\left[-\frac{U_X}{M}(1-\gamma)\theta+\beta J_X(M_X+1)J_C-\beta J_X J_{XC}\right][(M_X+1)J_Z+J_{CZ}]}{[(\theta-\theta\gamma)+(M_X+1)J_C-J_{CC}]}. \quad \text{(A21)}$$

Using (A21), we can evaluate M_Z.

(iv) Derivation of Consumption Function

To derive ELAST(c, x) in the DSGE consumption function (41), we eliminate x_{t+1} from (A1) using (A3). Under (29), this gives the percentage effect on consumption of a 1 per cent increase in real wealth at the start of year t ($x_t = 1$), holding Z_t and σ constant ($z_t = d\sigma = 0$). This is the definition of ELAST(c, x). Carrying out the elimination gives

$$\text{ELAST}(c,x) = \frac{[(M_X+1)J_X+J_{CX}-(1-\theta)(1-\gamma)]}{[(\theta-\theta\gamma)+(M_X+1)J_C-J_{CC}]}. \quad \text{(A22)}$$

The values of the coefficients on the right-hand side of (A22) are derived independently of the value of ρ. This confirms the assertion in Section VII that the introduction of a non-zero ρ does not affect ELAST(c, x).

ELAST(c, z) in the DSGE consumption function (41) is the percentage effect on consumption of a 1 per cent increase in Z ($z_t = 1$), holding X_t and σ constant ($x_t = d\sigma = 0$). This is what is revealed on the right-hand side of (A20):

$$\text{ELAST}(c,z) = \frac{\rho}{[(\theta-\theta\gamma)+(M_X+1)J_C-J_{CC}]}M_Z + \frac{(M_X+1)J_Z+J_{CZ}}{[(\theta-\theta\gamma)+(M_X+1)J_C-J_{CC}]}. \quad \text{(A23)}$$

If ρ is set to zero, then ELAST(c, z) is the value given by the z_t coefficient in (32). The introduction of a non-zero value for ρ leads to an extra term, confirming that shock persistence ($\rho > 0$) affects the value ELAST(c, z).