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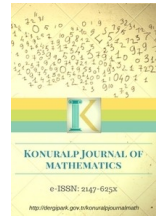
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This is the Published version of the following publication

Dragomir, Sever S (2023) Numerical Radius and  $p$ -Schatten Norm Inequalities for Analytic Functions of Operators in Hilbert Spaces. *Konuralp Journal of Mathematics*, 11 (2). pp. 109-126. ISSN 2147-625X

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# Numerical Radius and $p$ -Schatten Norm Inequalities for Analytic Functions of Operators in Hilbert Spaces

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## Abstract

Let  $H$  be a complex Hilbert space,  $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$  an analytic function on the domain  $G$  and  $A \in \mathcal{B}(H)$  with  $\text{Sp}(A) \subset G$  and  $\gamma$  a closed rectifiable path in  $G$  and such that  $\text{Sp}(A) \subset \text{ins}(\gamma)$ . If we denote

$$B(f, \gamma; A) := \frac{1}{2\pi} \int_{\gamma} |f(\xi)| (|\xi| - \|A\|)^{-1} |d\xi|,$$

then for  $B, C \in \mathcal{B}(H)$  we have

$$|\langle C^* A f(A) B x, y \rangle| \leq B(f, \gamma; A) \left\langle \|A\|^\alpha B^2 x, x \right\rangle^{1/2} \left\langle \|A^*\|^{1-\alpha} C^2 y, y \right\rangle^{1/2}$$

for  $\alpha \in [0, 1]$  and  $x, y \in H$ . Some natural applications for *numerical radius* and  *$p$ -Schatten norm* are also provided.

**Keywords:** Schwarz inequality, Vector inequality, Bounded operators, Numerical radius, Operator trace,  $p$ -Schatten norm.

**2010 Mathematics Subject Classification:** 47A63, 26D15, 46C05

## 1. Introduction

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [9]:

**Theorem 1.1.** Assume that  $h$  and  $g$  are non-negative functions on  $[0, \infty)$  which are continuous and satisfying the relation  $h(t)g(t) = t$  for all  $t \in [0, \infty)$ . For any  $T \in \mathcal{B}(H)$

$$|\langle Tx, y \rangle| \leq \|h(|T|)x\| \|g(|T^*|)y\| \quad (1.1)$$

for all  $x, y \in H$ .

If we take  $h(t) = t^\alpha$ ,  $g(t) = t^{1-\alpha}$  with  $\alpha \in [0, 1]$ , then we obtain Kato's inequality

$$|\langle Tx, y \rangle| \leq \| |T|^\alpha x \| \| |T^*|^{1-\alpha} y \| \quad \text{for all } x, y \in H. \quad (1.2)$$

The *numerical radius*  $w(T)$  of an operator  $T$  on  $H$  is given by

$$\omega(T) = \sup \{ |\langle Tx, x \rangle|, \|x\| = 1 \}. \quad (1.3)$$

Obviously, by (1.3), for any  $x \in H$  one has

$$|\langle Tx, x \rangle| \leq w(T) \|x\|^2. \quad (1.4)$$

It is well known that  $w(\cdot)$  is a norm on the Banach algebra  $\mathcal{B}(H)$  of all bounded linear operators  $T : H \rightarrow H$ , i.e.,

- (i)  $\omega(T) \geq 0$  for any  $T \in B(H)$  and  $\omega(T) = 0$  if and only if  $T = 0$ ;
- (ii)  $\omega(\lambda T) = |\lambda| \omega(T)$  for any  $\lambda \in \mathbb{C}$  and  $T \in B(H)$ ;
- (iii)  $\omega(T + V) \leq \omega(T) + \omega(V)$  for any  $T, V \in B(H)$ .

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \quad (1.5)$$

for any  $T \in B(H)$ .

F. Kittaneh, in 2003 [10], showed that for any operator  $T \in B(H)$  we have the following refinement of the first inequality in (1.5):

$$\omega(T) \leq \frac{1}{2} \left( \|T\| + \|T^2\|^{1/2} \right). \quad (1.6)$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [11] improved the inequality (1.5) as follows:

$$\frac{1}{4} \|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2} \|T^*T + TT^*\| \quad (1.7)$$

for any operator  $T \in B(H)$ .

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [8]:

If for an operator  $T \in B(H)$  we denote  $|T| := (T^*T)^{1/2}$ , then

$$\omega^r(T) \leq \frac{1}{2} \left\| |T|^{2\alpha r} + |T^*|^{2(1-\alpha)r} \right\| \quad (1.8)$$

and

$$\omega^{2r}(T) \leq \left\| \alpha |T|^{2r} + (1-\alpha) |T^*|^{2r} \right\|, \quad (1.9)$$

where  $\alpha \in (0, 1)$  and  $r \geq 1$ .

If we take  $\alpha = \frac{1}{2}$  and  $r = 1$  we get from (1.8) that

$$\omega(T) \leq \frac{1}{2} \left( \| |T| + |T^*| \| \right) \quad (1.10)$$

and from (1.9) that

$$\omega^2(T) \leq \frac{1}{2} \left( \| |T|^2 + |T^*|^2 \| \right). \quad (1.11)$$

For more related results, see the recent books on inequalities for numerical radii [2] and [6].

Let  $(H; \langle \cdot, \cdot \rangle)$  be a complex Hilbert space and  $\mathcal{B}(H)$  the Banach algebra of all bounded linear operators on  $H$ . If  $\{e_i\}_{i \in I}$  an orthonormal basis of  $H$ , we say that  $A \in \mathcal{B}(H)$  is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \quad (1.12)$$

The definition of  $\|A\|_1$  does not depend on the choice of the orthonormal basis  $\{e_i\}_{i \in I}$ . We denote by  $\mathcal{B}_1(H)$  the set of trace class operators in  $\mathcal{B}(H)$ .

We define the *trace* of a trace class operator  $A \in \mathcal{B}_1(H)$  to be

$$\text{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle, \quad (1.13)$$

where  $\{e_i\}_{i \in I}$  an orthonormal basis of  $H$ . Note that this coincides with the usual definition of the trace if  $H$  is finite-dimensional. We observe that the series (1.13) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

**Theorem 1.2.** *We have:*

(i) If  $A \in \mathcal{B}_1(H)$  then  $A^* \in \mathcal{B}_1(H)$  and

$$\text{tr}(A^*) = \overline{\text{tr}(A)}; \quad (1.14)$$

(ii) If  $A \in \mathcal{B}_1(H)$  and  $T \in \mathcal{B}(H)$ , then  $AT, TA \in \mathcal{B}_1(H)$  and

$$\text{tr}(AT) = \text{tr}(TA) \text{ and } |\text{tr}(AT)| \leq \|A\|_1 \|T\|; \quad (1.15)$$

(iii)  $\text{tr}(\cdot)$  is a bounded linear functional on  $\mathcal{B}_1(H)$  with  $\|\text{tr}\| = 1$ ;

(iv) If  $A, B \in \mathcal{B}_2(H)$  then  $AB, BA \in \mathcal{B}_1(H)$  and  $\text{tr}(AB) = \text{tr}(BA)$ ;

(v)  $\mathcal{B}_{fin}(H)$ , the space of operators of finite rank, is a dense subspace of  $\mathcal{B}_1(H)$ .

For a large number of results concerning trace inequalities, see the recent survey paper [7].

An operator  $A \in \mathcal{B}(H)$  is said to belong to the *von Neumann-Schatten class*  $\mathcal{B}_p(H)$ ,  $1 \leq p < \infty$  if the *p-Schatten norm* is finite [16, p. 60-64]

$$\|A\|_p := [\text{tr}(|A|^p)]^{1/p} = \left( \sum_{i \in I} \langle |A|^p e_i, e_i \rangle \right)^{1/p} < \infty.$$

For  $1 < p < q < \infty$  we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.16}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{1.17}$$

For  $p \geq 1$  the functional  $\|\cdot\|_p$  is a *norm* on the  $*$ -ideal  $\mathcal{B}_p(H)$  and  $(\mathcal{B}_p(H), \|\cdot\|_p)$  is a Banach space.

Also, see for instance [16, p. 60-64],

$$\|A\|_p = \|A^*\|_p, A \in \mathcal{B}_p(H) \tag{1.18}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, A, B \in \mathcal{B}_p(H) \tag{1.19}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \|BA\|_p \leq \|B\| \|A\|_p, A \in \mathcal{B}_p(H), B \in \mathcal{B}(H). \tag{1.20}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, A \in \mathcal{B}_p(H), B, C \in \mathcal{B}(H). \tag{1.21}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\text{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, A \in \mathcal{B}_p(H), B \in \mathcal{B}_q(H). \tag{1.22}$$

For the theory of trace functionals and their applications the reader is referred to [15] and [16].

For  $\mathcal{E} := \{e_i\}_{i \in I}$  an orthonormal basis of  $H$  we define for  $A \in \mathcal{B}_p(H)$ ,  $p \geq 1$

$$\|A\|_{\mathcal{E},p} := \left( \sum_{i \in I} |\langle Ae_i, e_i \rangle|^p \right)^{1/p}.$$

We observe that  $\|\cdot\|_{\mathcal{E},p}$  is a norm on  $\mathcal{B}_p(H)$  and

$$\|A\|_{\mathcal{E},p} \leq \|A\|_p \text{ for } A \in \mathcal{B}_p(H).$$

Further, we can take the supremum over all orthonormal basis in  $H$  we can also define, for  $A \in \mathcal{B}_p(H)$ , that

$$\omega_p(A) := \sup_{\mathcal{E}} \|A\|_{\mathcal{E},p} \leq \|A\|_p,$$

which is a *norm* on  $\mathcal{B}_p(H)$ .

It is also known that, if  $\mathcal{E} = \{e_i\}_{i \in I}$  and  $\mathcal{F} = \{f_i\}_{i \in I}$  are orthonormal basis, then [13]

$$\sup_{\mathcal{E}, \mathcal{F}} \sum_{i \in I} |\langle Te_i, f_i \rangle|^s = \|T\|_s^s \text{ for } s \geq 1. \tag{1.23}$$

Let  $\mathcal{B}$  be a unital Banach algebra,  $A \in \mathcal{B}$  and  $G$  be a domain of  $\mathbb{C}$  with  $\text{Sp}(A) \subset G$ . If  $f : G \rightarrow \mathbb{C}$  is analytic on  $G$ , we define an element  $f(A)$  in  $\mathcal{B}$  by

$$f(A) := \frac{1}{2\pi i} \int_{\delta} f(\xi) (\xi - A)^{-1} d\xi, \tag{1.24}$$

where  $\delta \subset G$  is taken to be closed rectifiable curve in  $G$  and such that  $\text{Sp}(A) \subset \text{ins}(\delta)$ , the inside of  $\delta$ .

It is well known (see for instance [4, pp. 201-204]) that  $f(A)$  does not depend on the choice of  $\delta$  and the *Spectral Mapping Theorem* (SMT)

$$\text{Sp}(f(A)) = f(\text{Sp}(A)) \tag{1.25}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [5] and [14].

## 2. Vector Inequalities

In 1988, F. Kittaneh [9, Corollary 7] obtained the following Schwarz type inequality for natural powers of operators:

**Lemma 2.1.** *Let  $A \in \mathcal{B}(H)$  and  $\alpha \in [0, 1]$ . Then for natural number  $n \geq 1$  we have*

$$|\langle A^n x, y \rangle|^2 \leq \|A\|^{2n-2} \langle |A|^{2\alpha} x, x \rangle \langle |A^*|^{2(1-\alpha)} y, y \rangle \quad (2.1)$$

for all  $x, y \in H$ .

We can state the following result as well:

**Corollary 2.2.** *Let  $A, B, C \in \mathcal{B}(H)$  and  $\alpha \in [0, 1]$ . Then for  $n \geq 1$  we have*

$$|\langle C^* A^n B x, y \rangle|^2 \leq \|A\|^{2n-2} \langle |A|^\alpha B|^2 x, x \rangle \langle |A^*|^{1-\alpha} C|^2 y, y \rangle \quad (2.2)$$

for all  $x, y \in H$ .

*Proof.* If we replace  $x$  by  $Bx$  and  $y$  by  $Cy$  in (2.1), then we get

$$|\langle C^* A^n B x, y \rangle|^2 \leq \|A\|^{2n-2} \langle B^* |A|^{2\alpha} B x, x \rangle \langle C^* |A^*|^{2(1-\alpha)} C y, y \rangle \quad (2.3)$$

for all  $x, y \in H$ .

Observe that  $B^* |A|^{2\alpha} B = \| |A|^\alpha B|^2$  and  $C^* |A^*|^{2(1-\alpha)} C = \| |A^*|^{1-\alpha} C|^2$ , then by (2.3) we get (2.2).  $\square$

We also have:

**Lemma 2.3.** *Assume that  $A, B, C \in \mathcal{B}(H)$  with  $\|A\| < 1$ , then*

$$\left| \langle C^* A (I-A)^{-1} B x, y \rangle \right| \leq (1 - \|A\|)^{-1} \times \langle |A|^\alpha B|^2 x, x \rangle^{1/2} \langle |A^*|^{1-\alpha} C|^2 y, y \rangle^{1/2} \quad (2.4)$$

for  $\alpha \in [0, 1]$  and  $x, y \in H$ . In particular,

$$\left| \langle C^* A (I-A)^{-1} B x, y \rangle \right| \leq (1 - \|A\|)^{-1} \times \langle |A|^{1/2} B|^2 x, x \rangle^{1/2} \langle |A^*|^{1/2} C|^2 y, y \rangle^{1/2} \quad (2.5)$$

for  $x, y \in H$ .

*Proof.* If we put  $n = k + 1$ ,  $k \in \mathbb{N}$  in (2.2) and take the square root, then we get

$$\left| \langle C^* A A^k B x, y \rangle \right| \leq \|A\|^k \langle |A|^\alpha B|^2 x, x \rangle^{1/2} \langle |A^*|^{1-\alpha} C|^2 y, y \rangle^{1/2}$$

for all  $x, y \in H$ .

Further, if we sum over  $k$  from 0 to  $m$ , then we obtain

$$\left| \left\langle C^* A \sum_{k=0}^m A^k B x, y \right\rangle \right| = \left| \sum_{k=0}^m \langle C^* A A^k B x, y \rangle \right| \leq \sum_{k=0}^m \left| \langle C^* A A^k B x, y \rangle \right| \leq \sum_{k=0}^m \|A\|^k \langle |A|^\alpha B|^2 x, x \rangle^{1/2} \langle |A^*|^{1-\alpha} C|^2 y, y \rangle^{1/2} \quad (2.6)$$

for all  $x, y \in H$ .

Since  $\|A\| < 1$ , then series  $\sum_{k=0}^{\infty} A^k$  and  $\sum_{k=0}^{\infty} \|A\|^k$  are convergent and

$$\sum_{k=0}^{\infty} A^k = (I - A)^{-1} \quad \text{and} \quad \sum_{k=0}^{\infty} \|A\|^k = (1 - \|A\|)^{-1}.$$

By taking now the limit over  $m \rightarrow \infty$  in (2.6) we deduce the desired result (2.4).  $\square$

Our first main result is as follows:

**Theorem 2.4.** *Let  $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$  be an analytic function on the domain  $G$  and  $A \in \mathcal{B}(H)$  with  $Sp(A) \subset G$  and  $\gamma$  a closed rectifiable path in  $G$  and such that  $Sp(A) \subset \text{ins}(\gamma)$ . If we denote*

$$B(f, \gamma; A) := \frac{1}{2\pi} \int_{\gamma} |f(\xi)| (|\xi| - \|A\|)^{-1} |d\xi|,$$

then for  $B, C \in \mathcal{B}(H)$  we have

$$|\langle C^* A f(A) B x, y \rangle| \leq B(f, \gamma; A) \langle |A|^\alpha B|^2 x, x \rangle^{1/2} \langle |A^*|^{1-\alpha} C|^2 y, y \rangle^{1/2} \quad (2.7)$$

for  $\alpha \in [0, 1]$  and  $x, y \in H$ . In particular,

$$\begin{aligned} & |\langle C^* A f(A) B x, y \rangle| \\ & \leq B(f, \gamma; A) \langle |A|^{1/2} B|^2 x, x \rangle^{1/2} \langle |A^*|^{1/2} C|^2 y, y \rangle^{1/2} \end{aligned} \quad (2.8)$$

for  $x, y \in H$ .

*Proof.* We have, by the representation

$$f(A) := \frac{1}{2\pi i} \int_{\gamma} f(\xi) (\xi I - A)^{-1} d\xi,$$

that

$$\begin{aligned} \langle C^* A f(A) Bx, y \rangle &= \left\langle C^* A \frac{1}{2\pi i} \int_{\gamma} f(\xi) (\xi I - A)^{-1} d\xi Bx, y \right\rangle \\ &= \frac{1}{2\pi i} \int_{\gamma} f(\xi) \langle C^* A (\xi I - A)^{-1} Bx, y \rangle d\xi \end{aligned}$$

for  $x, y \in H$ .

By taking the modulus and using the complex integral properties, we get

$$\begin{aligned} |\langle C^* A f(A) Bx, y \rangle| &\leq \frac{1}{2\pi} \int_{\gamma} |f(\xi)| \left| \langle C^* A (\xi I - A)^{-1} Bx, y \rangle \right| |d\xi| \\ &= \frac{1}{2\pi} \int_{\gamma} |f(\xi)| |\xi|^{-1} \left| \left\langle C^* A \left( I - \frac{A}{\xi} \right)^{-1} Bx, y \right\rangle \right| |d\xi| \end{aligned} \tag{2.9}$$

for  $x, y \in H$ .

Since  $\left\| \frac{A}{\xi} \right\| < 1$  for  $\xi \in \gamma$ , then by Lemma 2.3 for  $\frac{A}{\xi}$  we have

$$\begin{aligned} |\xi|^{-1} \left| \left\langle C^* A \left( I - \frac{A}{\xi} \right)^{-1} Bx, y \right\rangle \right| &= \left| \left\langle C^* \frac{A}{\xi} \left( I - \frac{A}{\xi} \right)^{-1} Bx, y \right\rangle \right| \leq \left( 1 - \left\| \frac{A}{\xi} \right\| \right)^{-1} \left\langle \left\| \frac{A}{\xi} \right\|^{\alpha} B \right|_{x,x} \right\rangle^{1/2} \left\langle \left\| \frac{A^*}{\bar{\xi}} \right\|^{1-\alpha} C \right|_{y,y} \right\rangle^{1/2} \\ &= \left( \frac{|\xi| - \|A\|}{|\xi|} \right)^{-1} \left\langle \left\| \frac{A}{\xi} \right\|^{\alpha} B \right|_{x,x} \right\rangle^{1/2} \left\langle \left\| \frac{A^*}{\bar{\xi}} \right\|^{1-\alpha} C \right|_{y,y} \right\rangle^{1/2} \\ &= \frac{|\xi|}{|\xi|^{\alpha} |\bar{\xi}|^{1-\alpha}} (|\xi| - \|A\|)^{-1} \left\langle |A|^{\alpha} B \right|_{x,x} \right\rangle^{1/2} \left\langle |A^*|^{1-\alpha} C \right|_{y,y} \right\rangle^{1/2} \\ &= (|\xi| - \|A\|)^{-1} \left\langle |A|^{\alpha} B \right|_{x,x} \right\rangle^{1/2} \left\langle |A^*|^{1-\alpha} C \right|_{y,y} \right\rangle^{1/2} \end{aligned} \tag{2.10}$$

for  $x, y \in H$ .

By utilizing (2.10) we derive

$$\frac{1}{2\pi} \int_{\gamma} |f(\xi)| |\xi|^{-1} \left| \left\langle C^* A \left( I - \frac{A}{\xi} \right)^{-1} Bx, y \right\rangle \right| |d\xi| \leq \left( \frac{1}{2\pi} \int_{\gamma} |f(\xi)| (|\xi| - \|A\|)^{-1} |d\xi| \right) \times \left\langle |A|^{\alpha} B \right|_{x,x} \right\rangle^{1/2} \left\langle |A^*|^{1-\alpha} C \right|_{y,y} \right\rangle^{1/2} \tag{2.11}$$

for  $x, y \in H$ .

By making use of (2.9) and (2.11) we obtain (2.7). □

**Remark 2.5.** For  $B = C = I$  in (2.7) we get the one operator inequalities

$$|\langle A f(A) x, y \rangle| \leq B(f, \gamma; A) \left\langle |A|^{2\alpha} \right|_{x,x} \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} \right|_{y,y} \right\rangle^{1/2} \tag{2.12}$$

for  $\alpha \in [0, 1]$  and  $x, y \in H$ . In particular,

$$|\langle A f(A) x, y \rangle| \leq B(f, \gamma; A) \langle |A| \right|_{x,x} \rangle^{1/2} \langle |A^*| \right|_{y,y} \rangle^{1/2} \tag{2.13}$$

for all  $x, y \in H$ .

If  $A$  is invertible and take  $C = I, B = A^{-1}$  in (2.7), then we get

$$|\langle f(A) x, y \rangle| \leq B(f, \gamma; A) \left\langle |A|^{-2(1-\alpha)} \right|_{x,x} \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} \right|_{y,y} \right\rangle^{1/2} \tag{2.14}$$

for  $\alpha \in [0, 1]$  and  $x, y \in H$ . In particular,

$$|\langle f(A) x, y \rangle| \leq B(f, \gamma; A) \left\langle |A|^{-1} \right|_{x,x} \right\rangle^{1/2} \langle |A^*| \right|_{y,y} \rangle^{1/2} \tag{2.15}$$

for  $x, y \in H$ .

If  $A > 0$  and we take  $B = A^{-\beta}, C = A^{-1+\beta}, \beta \in [0, 1]$ , then we derive

$$|\langle f(A) x, y \rangle| \leq B(f, \gamma; A) \left\langle A^{2(\alpha-\beta)} \right|_{x,x} \right\rangle^{1/2} \left\langle A^{2(\beta-\alpha)} \right|_{y,y} \right\rangle^{1/2} \tag{2.16}$$

for  $\alpha \in [0, 1]$  and  $x, y \in H$ . In particular, for  $\alpha = \beta$  we obtain

$$|\langle f(A) x, y \rangle| \leq B(f, \gamma; A) \|x\| \|y\| \tag{2.17}$$

for  $x, y \in H$ .

**Corollary 2.6.** *With the assumptions of Theorem 2.4 and if*

$$\|f\|_{\gamma,\infty} := \sup_{\xi \in \gamma} |f(\xi)| < \infty,$$

then, by denoting

$$B_\infty(f, \gamma; A) := \frac{1}{2\pi} \|f\|_{\gamma,\infty} \int_\gamma (|\xi| - \|A\|)^{-1} |d\xi|,$$

we have

$$\begin{aligned} & | \langle C^* A f(A) B x, y \rangle | \\ & \leq B_\infty(f, \gamma; A) \left\langle |A|^\alpha B|^2 x, x \right\rangle^{1/2} \left\langle |A^*|^{1-\alpha} C|^2 y, y \right\rangle^{1/2} \end{aligned} \tag{2.18}$$

for  $\alpha \in [0, 1]$  and  $x, y \in H$ . In particular,

$$\begin{aligned} & | \langle C^* A f(A) B x, y \rangle | \\ & \leq B_\infty(f, \gamma; A) \left\langle |A|^{1/2} B|^2 x, x \right\rangle^{1/2} \left\langle |A^*|^{1/2} C|^2 y, y \right\rangle^{1/2} \end{aligned} \tag{2.19}$$

for  $x, y \in H$ .

**Remark 2.7.** *If we assume that  $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$  is an analytic function on the domain  $G$  and  $A \in \mathcal{B}(H)$  with  $Sp(A) \subset D(0, R) \subset D$  where  $D(0, R)$  is an open disk centered in 0 and of radius  $R$ , then by taking  $\gamma$  parametrized by  $\xi(t) = Re^{2\pi it}$  where  $t \in [0, 1]$ , then  $d\xi(t) = 2\pi i Re^{2\pi it} dt$ ,  $|d\xi(t)| = 2\pi R dt$ ,  $|\xi| = R$  and by (2.18) we get for  $A, B \in \mathcal{B}(H)$  that*

$$| \langle C^* A f(A) B x, y \rangle | \leq \frac{R}{R - \|A\|} \int_0^1 |f(Re^{2\pi it})| dt \times \left\langle |A|^\alpha B|^2 x, x \right\rangle^{1/2} \left\langle |A^*|^{1-\alpha} C|^2 y, y \right\rangle^{1/2} \tag{2.20}$$

where  $\alpha \in [0, 1]$  and  $x, y \in H$ . In particular,

$$| \langle C^* A f(A) B x, y \rangle | \leq \frac{R}{R - \|A\|} \int_0^1 |f(Re^{2\pi it})| dt \times \left\langle |A|^{1/2} B|^2 x, x \right\rangle^{1/2} \left\langle |A^*|^{1/2} C|^2 y, y \right\rangle^{1/2} \tag{2.21}$$

for  $x, y \in H$ .

Moreover, if  $\|f\|_{R,\infty} := \sup_{t \in [0,1]} |f(Re^{2\pi it})| < \infty$ , then we have the simpler inequalities

$$| \langle C^* A f(A) B x, y \rangle | \leq \frac{R \|f\|_{R,\infty}}{R - \|A\|} \left\langle |A|^\alpha B|^2 x, x \right\rangle^{1/2} \left\langle |A^*|^{1-\alpha} C|^2 y, y \right\rangle^{1/2} \tag{2.22}$$

for  $x, y \in H$ . In particular,

$$| \langle C^* A f(A) B x, y \rangle | \leq \frac{R \|f\|_{R,\infty}}{R - \|A\|} \times \left\langle |A|^{1/2} B|^2 x, x \right\rangle^{1/2} \left\langle |A^*|^{1/2} C|^2 y, y \right\rangle^{1/2} \tag{2.23}$$

for  $x, y \in H$ .

### 3. Norm and Numerical Radius Inequalities

The following vector inequality for positive operators  $A \geq 0$ , obtained by C. A. McCarthy in [12] is well known,

$$\langle Ax, x \rangle^p \leq \langle A^p x, x \rangle, \quad p \geq 1$$

for  $x \in H, \|x\| = 1$ .

Buzano's inequality [3],

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y \rangle|] \geq |\langle x, e \rangle \langle e, y \rangle| \tag{3.1}$$

that holds for any  $x, y, e \in H$  with  $\|e\| = 1$  will also be used in the sequel.

We also have the following norm and numerical radius inequalities:

**Theorem 3.1.** *Let  $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$  be an analytic function on the domain  $G$  and  $A \in \mathcal{B}(H)$  with  $Sp(A) \subset G$  and  $\gamma$  a closed rectifiable path in  $G$  and such that  $Sp(A) \subset \text{ins}(\gamma)$ . If  $B, C \in \mathcal{B}(H)$ , then we have the norm inequality*

$$\|C^* A f(A) B\| \leq B(f, \gamma; A) \| |A|^\alpha B \| \| |A^*|^{1-\alpha} C \|. \tag{3.2}$$

We also have the numerical radius inequalities

$$\omega(C^* A f(A) B) \leq \frac{1}{2} B(f, \gamma; A) \left\| |A|^\alpha B|^2 + |A^*|^{1-\alpha} C|^2 \right\| \tag{3.3}$$

and

$$\begin{aligned} & \omega^2(C^* A f(A) B) \\ & \leq \frac{1}{2} B^2(f, \gamma; A) \left[ \| |A|^\alpha B \|^2 \| |A^*|^{1-\alpha} C \|^2 + \omega \left( |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 \right) \right]. \end{aligned} \tag{3.4}$$

*Proof.* We have from (2.7), by taking the supremum over  $\|x\| = \|y\| = 1$ , that

$$\begin{aligned} \|C^*Af(A)B\|^2 &= \sup_{\|x\|=\|y\|=1} |\langle C^*Af(A)Bx,y \rangle|^2 \\ &\leq B^2(f,\gamma;A) \sup_{\|x\|=1} \langle |A|^\alpha B|^2 x,x \rangle \sup_{\|y\|=1} \langle |A^*|^{1-\alpha} C|^2 y,y \rangle \\ &= B^2(f,\gamma;A) \left\| |A|^\alpha B|^2 \right\| \left\| |A^*|^{1-\alpha} C|^2 \right\| \\ &= B^2(f,\gamma;A) \left\| |A|^\alpha B|^2 \right\| \left\| |A^*|^{1-\alpha} C|^2 \right\|^2, \end{aligned}$$

which gives (3.2).

From (2.7) we get, by taking  $y = x$ , the square root and using the *A-G-mean inequality*, that

$$\begin{aligned} |\langle C^*Af(A)Bx,x \rangle| &\leq B(f,\gamma;A) \langle |A|^\alpha B|^2 x,x \rangle^{1/2} \langle |A^*|^{1-\alpha} C|^2 x,x \rangle^{1/2} \\ &\leq \frac{1}{2} B(f,\gamma;A) \left( \langle |A|^\alpha B|^2 x,x \rangle + \langle |A^*|^{1-\alpha} C|^2 x,x \rangle \right) \\ &= \frac{1}{2} B(f,\gamma;A) \left\langle \left( |A|^\alpha B|^2 + |A^*|^{1-\alpha} C|^2 \right) x,x \right\rangle \end{aligned} \tag{3.5}$$

for all  $x \in H$ .

By taking the supremum over  $\|x\| = 1$  in (3.5) we get that

$$\begin{aligned} \omega(C^*Af(A)B) &= \sup_{\|x\|=1} |\langle C^*Af(A)Bx,x \rangle| \\ &\leq \frac{1}{2} B(f,\gamma;A) \sup_{\|x\|=1} \left\langle \left( |A|^\alpha B|^2 + |A^*|^{1-\alpha} C|^2 \right) x,x \right\rangle \\ &= \frac{1}{2} B(f,\gamma;A) \left\| |A|^\alpha B|^2 + |A^*|^{1-\alpha} C|^2 \right\|, \end{aligned}$$

which proves (3.2).

From (2.7) for  $y = x$  and Buzano's inequality we derive that

$$\begin{aligned} |\langle C^*Af(A)Bx,x \rangle|^2 &\leq B^2(f,\gamma;A) \langle |A|^\alpha B|^2 x,x \rangle \langle x, |A^*|^{1-\alpha} C|^2 x \rangle \\ &\leq \frac{1}{2} B^2(f,\gamma;A) \\ &\quad \times \left[ \left\| |A|^\alpha B|^2 x \right\| \left\| |A^*|^{1-\alpha} C|^2 x \right\| + \left| \langle |A|^\alpha B|^2 x, |A^*|^{1-\alpha} C|^2 x \rangle \right| \right] \\ &= \frac{1}{2} B^2(f,\gamma;A) \\ &\quad \times \left[ \left\| |A|^\alpha B|^2 x \right\| \left\| |A^*|^{1-\alpha} C|^2 x \right\| + \left| \langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 x,x \rangle \right| \right] \end{aligned} \tag{3.6}$$

for all  $x \in H$ .

By taking the supremum over  $\|x\| = 1$  in (3.6) we get that

$$\begin{aligned} \omega^2(C^*Af(A)B) &= \sup_{\|x\|=1} |\langle C^*Af(A)Bx,x \rangle|^2 \\ &\leq \frac{1}{2} B^2(f,\gamma;A) \times \sup_{\|x\|=1} \left[ \left\| |A|^\alpha B|^2 x \right\| \left\| |A^*|^{1-\alpha} C|^2 x \right\| + \left| \langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 x,x \rangle \right| \right] \\ &\leq \frac{1}{2} B^2(f,\gamma;A) \times \left[ \sup_{\|x\|=1} \left\{ \left\| |A|^\alpha B|^2 x \right\| \left\| |A^*|^{1-\alpha} C|^2 x \right\| \right\} + \sup_{\|x\|=1} \left| \langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 x,x \rangle \right| \right] \\ &\leq \frac{1}{2} B^2(f,\gamma;A) \times \left[ \sup_{\|x\|=1} \left\| |A|^\alpha B|^2 x \right\| \sup_{\|x\|=1} \left\| |A^*|^{1-\alpha} C|^2 x \right\| + \sup_{\|x\|=1} \left| \langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 x,x \rangle \right| \right] \\ &= \frac{1}{2} B^2(f,\gamma;A) \left[ \left\| |A|^\alpha B|^2 \right\| \left\| |A^*|^{1-\alpha} C|^2 \right\| + \omega \left( |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 \right) \right] \\ &= \frac{1}{2} B^2(f,\gamma;A) \left[ \left\| |A|^\alpha B|^2 \right\| \left\| |A^*|^{1-\alpha} C|^2 \right\|^2 + \omega \left( |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 \right) \right], \end{aligned}$$

which proves (3.4). □

**Remark 3.2.** If we take  $\alpha = 1/2$  in Theorem 3.1, then we get the norm inequality

$$\|C^*Af(A)B\| \leq B(f, \gamma; A) \left\| |A|^{1/2} B \right\| \left\| |A^*|^{1/2} C \right\| \quad (3.7)$$

and the numerical radius inequalities

$$\omega(C^*Af(A)B) \leq \frac{1}{2} B(f, \gamma; A) \left\| |A|^{1/2} B \right\|^2 + \left\| |A^*|^{1/2} C \right\|^2 \quad (3.8)$$

and

$$\begin{aligned} \omega^2(C^*Af(A)B) & \leq \frac{1}{2} B^2(f, \gamma; A) \left[ \left\| |A|^{1/2} B \right\|^2 \left\| |A^*|^{1/2} C \right\|^2 + \omega \left( \left\| |A^*|^{1/2} C \right\|^2 \left\| |A|^{1/2} B \right\|^2 \right) \right]. \end{aligned} \quad (3.9)$$

The second main result is as follows:

**Theorem 3.3.** Assume that the conditions of Theorem 3.1 are satisfied. If  $\alpha \in [0, 1]$ ,  $r > 0$ ,  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  and  $pr, qr \geq 1$ , then

$$\omega^{2r}(C^*Af(A)B) \leq B^{2r}(f, \gamma; A) \left\| \frac{1}{p} |A|^\alpha B^{2rp} + \frac{1}{q} |A^*|^{1-\alpha} C^{2rq} \right\|. \quad (3.10)$$

If  $r \geq 1$ , then

$$\omega^{2r}(C^*Af(A)B) \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[ \left\| |A|^\alpha B \right\|^{2r} \left\| |A^*|^{1-\alpha} C \right\|^{2r} + \omega^r \left( \left\| |A^*|^{1-\alpha} C \right\|^2 \left\| |A|^\alpha B \right\|^2 \right) \right]. \quad (3.11)$$

If  $r \geq 1$ ,  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  and  $pr, qr \geq 2$ , then also

$$\omega^{2r}(C^*Af(A)B) \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left( \left\| \frac{1}{p} |A|^\alpha B^{2rp} + \frac{1}{q} |A^*|^{1-\alpha} C^{2rq} \right\| + \omega^r \left( \left\| |A^*|^{1-\alpha} C \right\|^2 \left\| |A|^\alpha B \right\|^2 \right) \right). \quad (3.12)$$

*Proof.* If we take the power  $r > 0$  in (2.7) written for  $y = x$  then we get, by Young and McCarthy inequalities that

$$\begin{aligned} |\langle C^*Af(A)Bx, x \rangle|^{2r} & \leq B^{2r}(f, \gamma; A) \left\langle |A|^\alpha B^2 x, x \right\rangle^r \left\langle |A^*|^{1-\alpha} C^2 x, x \right\rangle^r \\ & \leq B^{2r}(f, \gamma; A) \left[ \frac{1}{p} \left\langle |A|^\alpha B^2 x, x \right\rangle^{rp} + \frac{1}{q} \left\langle |A^*|^{1-\alpha} C^2 x, x \right\rangle^{rq} \right] \\ & \leq B^{2r}(f, \gamma; A) \left[ \frac{1}{p} \left\langle |A|^\alpha B^{2rp} x, x \right\rangle + \frac{1}{q} \left\langle |A^*|^{1-\alpha} C^{2rq} x, x \right\rangle \right] \\ & = B^{2r}(f, \gamma; A) \left[ \left\langle \frac{1}{p} |A|^\alpha B^{2rp} + \frac{1}{q} |A^*|^{1-\alpha} C^{2rq} x, x \right\rangle \right] \end{aligned}$$

for  $x \in H$  with  $\|x\| = 1$ .

By taking the supremum over  $\|x\| = 1$ , then we get that

$$\begin{aligned} \omega^{2r}(C^*Af(A)B) & = \sup_{\|x\|=1} |\langle C^*Af(A)Bx, x \rangle|^{2r} \\ & \leq B^{2r}(f, \gamma; A) \sup_{\|x\|=1} \left[ \left\langle \left( \frac{1}{p} |A|^\alpha B^{2rp} + \frac{1}{q} |A^*|^{1-\alpha} C^{2rq} \right) x, x \right\rangle \right] \\ & = B^{2r}(f, \gamma; A) \left\| \frac{1}{p} |A|^\alpha B^{2rp} + \frac{1}{q} |A^*|^{1-\alpha} C^{2rq} \right\|, \end{aligned}$$

which proves (3.10).

If we take the power  $r \geq 1$  in (3.6) and by using the convexity of the power function, we get

$$\begin{aligned} |\langle C^*Af(A)Bx, x \rangle|^{2r} & = B^{2r}(f, \gamma; A) \times \left[ \frac{\left\| |A|^\alpha B^2 x \right\|^r \left\| |A^*|^{1-\alpha} C^2 x \right\|^r + \left\langle |A^*|^{1-\alpha} C^2 |A|^\alpha B^2 x, x \right\rangle^r}{2} \right]^r \\ & \leq B^{2r}(f, \gamma; A) \times \frac{\left\| |A|^\alpha B^2 x \right\|^r \left\| |A^*|^{1-\alpha} C^2 x \right\|^r + \left\langle |A^*|^{1-\alpha} C^2 |A|^\alpha B^2 x, x \right\rangle^r}{2} \end{aligned} \quad (3.13)$$

for  $x \in H$  with  $\|x\| = 1$ .

By taking the supremum over  $\|x\| = 1$ , then we get that

$$\begin{aligned} \omega^{2r}(C^*Af(A)B) & \leq B^{2r}(f, \gamma; A) \times \frac{\left\| |A|^\alpha B \right\|^{2r} \left\| |A^*|^{1-\alpha} C \right\|^{2r} + \omega^r \left( \left\| |A^*|^{1-\alpha} C \right\|^2 \left\| |A|^\alpha B \right\|^2 \right)}{2} \\ & = B^{2r}(f, \gamma; A) \times \frac{\left\| |A|^\alpha B \right\|^{2r} \left\| |A^*|^{1-\alpha} C \right\|^{2r} + \omega^r \left( \left\| |A^*|^{1-\alpha} C \right\|^2 \left\| |A|^\alpha B \right\|^2 \right)}{2}, \end{aligned}$$

which proves (3.11).

Also, observe that

$$\begin{aligned} \left\| |A|^\alpha B^2 x \right\|^r \left\| |A^*|^{1-\alpha} C^2 x \right\|^r &\leq \frac{1}{p} \left\| |A|^\alpha B^2 x \right\|^{pr} + \frac{1}{q} \left\| |A^*|^{1-\alpha} C^2 x \right\|^{qr} \\ &= \frac{1}{p} \left\| |A|^\alpha B^2 x \right\|^{2\frac{pr}{2}} + \frac{1}{q} \left\| |A^*|^{1-\alpha} C^2 x \right\|^{2\frac{qr}{2}} \\ &= \frac{1}{p} \left\langle |A|^\alpha B^4 x, x \right\rangle^{\frac{pr}{2}} + \frac{1}{q} \left\langle |A^*|^{1-\alpha} C^4 x, x \right\rangle^{\frac{qr}{2}} \\ &\leq \frac{1}{p} \left\langle |A|^\alpha B^{2pr} x, x \right\rangle + \frac{1}{q} \left\langle |A^*|^{1-\alpha} C^{2qr} x, x \right\rangle \\ &= \left\langle \left( \frac{1}{p} |A|^\alpha B^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C^{2qr} \right) x, x \right\rangle, \end{aligned}$$

for  $x \in H$  with  $\|x\| = 1$ . Then

$$\frac{\left\| |A|^\alpha B^2 x \right\|^r \left\| |A^*|^{1-\alpha} C^2 x \right\|^r + \left| \left\langle |A^*|^{1-\alpha} C^2 |A|^\alpha B^2 x, x \right\rangle \right|^r}{2} \leq \frac{1}{2} \left[ \left\langle \left( \frac{1}{p} |A|^\alpha B^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C^{2qr} \right) x, x \right\rangle + \left| \left\langle |A^*|^{1-\alpha} C^2 |A|^\alpha B^2 x, x \right\rangle \right|^r \right]$$

and by (3.13)

$$|\langle C^* A f(A) B x, x \rangle|^{2r} \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[ \left\langle \left( \frac{1}{p} |A|^\alpha B^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C^{2qr} \right) x, x \right\rangle + \left| \left\langle |A^*|^{1-\alpha} C^2 |A|^\alpha B^2 x, x \right\rangle \right|^r \right]$$

for  $x \in H$  with  $\|x\| = 1$ .

By taking the supremum over  $\|x\| = 1$ , we derive (3.12). □

**Remark 3.4.** If we take  $r = 1$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  in (3.10), then we obtain

$$\omega^2(C^* A f(A) B) \leq B^2(f, \gamma; A) \left\| \frac{1}{p} |A|^\alpha B^{2p} + \frac{1}{q} |A^*|^{1-\alpha} C^{2q} \right\|, \tag{3.14}$$

which for  $p = q = 2$  gives

$$\omega^2(C^* A f(A) B) \leq \frac{1}{2} B^2(f, \gamma; A) \left\| |A|^\alpha B^4 + |A^*|^{1-\alpha} C^4 \right\|. \tag{3.15}$$

If we take  $r = 1$  and  $p = q = 2$  in (3.12), then we get

$$\omega^2(C^* A f(A) B) \leq \frac{1}{2} B^2(f, \gamma; A) \left( \frac{1}{2} \left\| |A|^\alpha B^4 + |A^*|^{1-\alpha} C^4 \right\| + \omega \left( |A^*|^{1-\alpha} C^2 |A|^\alpha B^2 \right) \right). \tag{3.16}$$

If we take  $r = 2$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  in (3.12), then we get

$$\omega^4(C^* A f(A) B) \leq \frac{1}{2} B^4(f, \gamma; A) \left( \left\| \frac{1}{p} |A|^\alpha B^{4p} + \frac{1}{q} |A^*|^{1-\alpha} C^{4q} \right\| + \omega^2 \left( |A^*|^{1-\alpha} C^2 |A|^\alpha B^2 \right) \right). \tag{3.17}$$

We also have:

**Theorem 3.5.** With the assumptions of Theorem 3.1, we have for  $r \geq 1, \lambda \in [0, 1]$  that

$$\omega^2(C^* A f(A) B) \leq B^2(f, \gamma; A) \left\| (1 - \lambda) |A|^\alpha B^{2r} + \lambda |A^*|^{1-\alpha} C^{2r} \right\|^{1/r} \times \| |A|^\alpha B \|^{2\lambda} \| |A^*|^{1-\alpha} C \|^{2(1-\lambda)} \tag{3.18}$$

for all  $\alpha \in [0, 1]$ .

Also, we have

$$\omega^2(C^* A f(A) B) \leq B^2(f, \gamma; A) \left\| (1 - \lambda) |A|^\alpha B^{2r} + \lambda |A^*|^{1-\alpha} C^{2r} \right\|^{1/r} \times \left\| \lambda |A|^\alpha B^{2r} + (1 - \lambda) |A^*|^{1-\alpha} C^{2r} \right\|^{1/r} \tag{3.19}$$

for all  $\alpha \in [0, 1]$  and  $r \geq 1$ .

*Proof.* From the first part of (3.6) we have

$$\begin{aligned} |\langle C^*Af(A)Bx, x \rangle|^2 &\leq B^2(f, \gamma; A) \langle |A|^\alpha B|^2 x, x \rangle \langle x, |A^*|^{1-\alpha} C|^2 x \rangle \\ &= B^2(f, \gamma; A) \langle |A|^\alpha B|^2 x, x \rangle^{1-\lambda} \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^\lambda \times \langle |A|^\alpha B|^2 x, x \rangle^\lambda \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^{1-\lambda} \\ &\leq B^2(f, \gamma; A) \left[ (1-\lambda) \langle |A|^\alpha B|^2 x, x \rangle + \lambda \langle x, |A^*|^{1-\alpha} C|^2 x \rangle \right] \times \langle |A|^\alpha B|^2 x, x \rangle^\lambda \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^{1-\lambda} \end{aligned}$$

for all  $x \in H$ ,  $\|x\| = 1$ .

If we take the power  $r \geq 1$ , then we get by the convexity of power  $r$  that

$$\begin{aligned} |\langle C^*Af(A)Bx, x \rangle|^{2r} &\leq B^{2r}(f, \gamma; A) \left[ (1-\lambda) \langle |A|^\alpha B|^2 x, x \rangle + \lambda \langle x, |A^*|^{1-\alpha} C|^2 x \rangle \right]^r \\ &\quad \times \langle |A|^\alpha B|^2 x, x \rangle^{r\lambda} \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^{r(1-\lambda)} \\ &\leq B^{2r}(f, \gamma; A) \left[ (1-\lambda) \langle |A|^\alpha B|^2 x, x \rangle^r + \lambda \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^r \right] \\ &\quad \times \langle |A|^\alpha B|^2 x, x \rangle^{r\lambda} \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^{r(1-\lambda)} \end{aligned} \quad (3.20)$$

for all  $x \in H$ ,  $\|x\| = 1$ .

If we use McCarthy inequality for power  $r \geq 1$ , then we get

$$\begin{aligned} (1-\lambda) \langle |A|^\alpha B|^2 x, x \rangle^r + \lambda \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^r &\leq (1-\lambda) \langle |A|^\alpha B|^{2r} x, x \rangle + \lambda \langle x, |A^*|^{1-\alpha} C|^{2r} x \rangle \\ &= \left\langle \left[ (1-\lambda) |A|^\alpha B|^{2r} + \lambda |A^*|^{1-\alpha} C|^{2r} \right] x, x \right\rangle \end{aligned}$$

and by (3.20)

$$|\langle C^*Af(A)Bx, x \rangle|^{2r} \leq B^{2r}(f, \gamma; A) \left[ \left\langle \left[ (1-\lambda) |A|^\alpha B|^{2r} + \lambda |A^*|^{1-\alpha} C|^{2r} \right] x, x \right\rangle \right] \times \langle |A|^\alpha B|^2 x, x \rangle^{r\lambda} \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^{r(1-\lambda)} \quad (3.21)$$

for all  $x \in H$ ,  $\|x\| = 1$ .

If we take the supremum over  $\|x\| = 1$ , then we get

$$\begin{aligned} \omega^{2r}(C^*Af(A)B) &= \sup_{\|x\|=1} |\langle C^*Af(A)Bx, x \rangle|^{2r} \\ &\leq B^{2r}(f, \gamma; A) \sup_{\|x\|=1} \left[ \left\langle \left[ (1-\lambda) |A|^\alpha B|^{2r} + \lambda |A^*|^{1-\alpha} C|^{2r} \right] x, x \right\rangle \right] \\ &\quad \times \sup_{\|x\|=1} \langle |A|^\alpha B|^2 x, x \rangle^{r\lambda} \sup_{\|x\|=1} \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^{r(1-\lambda)} \\ &= B^{2r}(f, \gamma; A) \left\| (1-\lambda) |A|^\alpha B|^{2r} + \lambda |A^*|^{1-\alpha} C|^{2r} \right\| \times \| |A|^\alpha B \|^{2r\lambda} \| |A^*|^{1-\alpha} C \|^{2r(1-\lambda)}, \end{aligned}$$

which gives (3.18).

We also have

$$|\langle C^*Af(A)Bx, x \rangle|^{2r} \leq B^{2r}(f, \gamma; A) \left[ \left\langle \left[ (1-\lambda) |A|^\alpha B|^{2r} + \lambda |A^*|^{1-\alpha} C|^{2r} \right] x, x \right\rangle \right] \times \left[ \left\langle \left[ \lambda |A|^\alpha B|^{2r} + (1-\lambda) |A^*|^{1-\alpha} C|^{2r} \right] x, x \right\rangle \right]$$

for all  $x \in H$ ,  $\|x\| = 1$ , which proves (3.19).  $\square$

**Remark 3.6.** If we take  $r = 1$  in Theorem 3.5, then we get

$$\omega^2(C^*Af(A)B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) |A|^\alpha B|^2 + \lambda |A^*|^{1-\alpha} C|^2 \right\| \times \| |A|^\alpha B \|^{2\lambda} \| |A^*|^{1-\alpha} C \|^{2(1-\lambda)} \quad (3.22)$$

and

$$\omega^2(C^*Af(A)B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) |A|^\alpha B|^2 + \lambda |A^*|^{1-\alpha} C|^2 \right\| \times \left\| \lambda |A|^\alpha B|^2 + (1-\lambda) |A^*|^{1-\alpha} C|^2 \right\| \quad (3.23)$$

for all  $\alpha, \lambda \in [0, 1]$ .

If we take  $\lambda = 1/2$  in (3.22), then we obtain

$$\omega^2(C^*Af(A)B) \leq \frac{1}{2} B^2(f, \gamma; A) \left\| |A|^\alpha B|^2 + |A^*|^{1-\alpha} C|^2 \right\| \| |A|^\alpha B \| \| |A^*|^{1-\alpha} C \|. \quad (3.24)$$

If we take  $r = 2$  in Theorem 3.5, then we get

$$\omega^2(C^*Af(A)B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) |A|^\alpha B^4 + \lambda |A^*|^{1-\alpha} C^4 \right\|^{1/2} \times \left\| |A|^\alpha B \right\|^{2\lambda} \left\| |A^*|^{1-\alpha} C \right\|^{2(1-\lambda)} \tag{3.25}$$

and

$$\omega^2(C^*Af(A)B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) |A|^\alpha B^4 + \lambda |A^*|^{1-\alpha} C^4 \right\|^{1/2} \times \left\| \lambda |A|^\alpha B^4 + (1-\lambda) |A^*|^{1-\alpha} C^4 \right\|^{1/2} \tag{3.26}$$

for all  $\alpha, \lambda \in [0, 1]$ .

If we take  $\lambda = 1/2$  in (3.25), then we obtain

$$\begin{aligned} \omega^2(C^*Af(A)B) & \\ \leq \frac{\sqrt{2}}{2} B^2(f, \gamma; A) & \left\| |A|^\alpha B^4 + |A^*|^{1-\alpha} C^4 \right\|^{1/2} \left\| |A|^\alpha B \right\| \left\| |A^*|^{1-\alpha} C \right\|. \end{aligned} \tag{3.27}$$

### 4. Inequalities for Trace of Operators

We have the following result for trace of operators:

**Theorem 4.1.** Let  $r \geq 1/2, p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  and  $pr, qr \geq 1$ . Let  $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$  be an analytic function on the domain  $G$  and  $A \in \mathcal{B}(H)$  with  $Sp(A) \subset G$  and  $\gamma$  a closed rectifiable path in  $G$  and such that  $Sp(A) \subset ins(\gamma)$ . If  $B, C \in \mathcal{B}(H)$  with  $|A|^\alpha B \in \mathcal{B}_{2pr}(H)$  and  $|A^*|^{1-\alpha} C \in \mathcal{B}_{2qr}(H)$  for  $\alpha \in [0, 1]$ , then  $C^*Af(A)B \in \mathcal{B}_{2r}(H)$  and

$$\|C^*Af(A)B\|_{2r} \leq B(f, \gamma; A) \left\| |A|^\alpha B \right\|_{2pr} \left\| |A^*|^{1-\alpha} C \right\|_{2qr}. \tag{4.1}$$

In particular,

$$\|C^*Af(A)B\|_{2r} \leq B(f, \gamma; A) \left\| |A|^{1/2} B \right\|_{2pr} \left\| |A^*|^{1/2} C \right\|_{2qr} \tag{4.2}$$

for  $|A|^{1/2} B \in \mathcal{B}_{2pr}(H)$  and  $|A^*|^{1/2} C \in \mathcal{B}_{2qr}(H)$ .

*Proof.* If we take in (2.7) the power  $r > 0$  and  $x = e_i, y = f_i$  where  $\mathcal{E} = \{e_i\}_{i \in I}$  and  $\mathcal{F} = \{f_i\}_{i \in I}$  are orthonormal basis and sum, then we get

$$\sum_{i \in I} |(C^*Af(A)Be_i, f_i)|^{2r} \leq B^{2r}(f, \gamma; A) \sum_{i \in I} \left\langle |A|^\alpha B^2 e_i, e_i \right\rangle^r \left\langle |A^*|^{1-\alpha} C^2 f_i, f_i \right\rangle^r. \tag{4.3}$$

If we use the Hölder's inequality for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then we get

$$\sum_{i \in I} \left\langle |A|^\alpha B^2 e_i, e_i \right\rangle^r \left\langle |A^*|^{1-\alpha} C^2 f_i, f_i \right\rangle^r \leq \left( \sum_{i \in I} \left\langle |A|^\alpha B^2 e_i, e_i \right\rangle^{pr} \right)^{1/p} \left( \sum_{i \in I} \left\langle |A^*|^{1-\alpha} C^2 f_i, f_i \right\rangle^{qr} \right)^{1/q}. \tag{4.4}$$

By the McCarthy inequality for  $pr, qr \geq 1$ , we have

$$\sum_{i \in I} \left\langle |A|^\alpha B^2 e_i, e_i \right\rangle^{pr} \leq \sum_{i \in I} \left\langle |A|^\alpha B^{2pr} e_i, e_i \right\rangle$$

and

$$\sum_{i \in I} \left\langle |A^*|^{1-\alpha} C^2 f_i, f_i \right\rangle^{qr} \leq \sum_{i \in I} \left\langle |A^*|^{1-\alpha} C^{2qr} f_i, f_i \right\rangle,$$

therefore

$$\begin{aligned} \left( \sum_{i \in I} \left\langle |A|^\alpha B^2 e_i, e_i \right\rangle^{pr} \right)^{1/p} \left( \sum_{i \in I} \left\langle |A^*|^{1-\alpha} C^2 f_i, f_i \right\rangle^{qr} \right)^{1/q} & \leq \left( \sum_{i \in I} \left\langle |A|^\alpha B^{2pr} e_i, e_i \right\rangle \right)^{1/p} \left( \sum_{i \in I} \left\langle |A^*|^{1-\alpha} C^{2qr} f_i, f_i \right\rangle \right)^{1/q} \\ & = \left( \| |A|^\alpha B \|_{2pr}^{2pr} \right)^{1/p} \left( \| |A^*|^{1-\alpha} C \|_{2qr}^{2qr} \right)^{1/q} = \| |A|^\alpha B \|_{2pr}^{2r} \| |A^*|^{1-\alpha} C \|_{2qr}^{2r}. \end{aligned}$$

By (4.3) and (4.4) we derive

$$\sum_{i \in I} |(C^*Af(A)Be_i, f_i)|^{2r} \leq B^{2r}(f, \gamma; A) \left\| |A|^\alpha B \right\|_{2pr}^{2r} \left\| |A^*|^{1-\alpha} C \right\|_{2qr}^{2r}. \tag{4.5}$$

Now, if we take the supremum over  $\mathcal{E}$  and  $\mathcal{F}$  in (4.5), then by (1.23) we get

$$\|C^*Af(A)B\|_{2r}^{2r} \leq B^{2r}(f, \gamma; A) \left\| |A|^\alpha B \right\|_{2pr}^{2r} \left\| |A^*|^{1-\alpha} C \right\|_{2qr}^{2r}$$

and the inequality (4.1) is obtained. □

**Remark 4.2.** If we take  $r = 1/2$  and  $p = q = 2$ , then by (4.1) we get

$$\|C^*Af(A)B\|_1 \leq B(f, \gamma; A) \| |A|^\alpha B \|_2 \| |A^*|^{1-\alpha} C \|_2 \quad (4.6)$$

provided that  $|A|^\alpha B \in \mathcal{B}_2(H)$  and  $|A^*|^{1-\alpha} C \in \mathcal{B}_2(H)$  for  $\alpha \in [0, 1]$ .  
Also, if  $r = 1$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then by (4.1) we get

$$\|C^*Af(A)B\|_2 \leq B(f, \gamma; A) \| |A|^\alpha B \|_{2p} \| |A^*|^{1-\alpha} C \|_{2q} \quad (4.7)$$

provided that  $|A|^\alpha B \in \mathcal{B}_{2p}(H)$  and  $|A^*|^{1-\alpha} C \in \mathcal{B}_{2q}(H)$  for  $\alpha \in [0, 1]$ .

We also have:

**Theorem 4.3.** Let  $r \geq 1/2$ ,  $p, q \geq 1$  with  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ . Let  $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$  be an analytic function on the domain  $G$  and  $A \in \mathcal{B}(H)$  with  $Sp(A) \subset G$  and  $\gamma$  a closed rectifiable path in  $G$  and such that  $Sp(A) \subset \text{ins}(\gamma)$ . If  $|A|^\alpha B \in \mathcal{B}_{2p}(H)$  and  $|A^*|^{1-\alpha} C \in \mathcal{B}_{2q}(H)$  for  $\alpha \in [0, 1]$ , then  $C^*Af(A)B \in \mathcal{B}_{2r}(H)$  and

$$\|C^*Af(A)B\|_{2r} \leq B(f, \gamma; A) \| |A|^\alpha B \|_{2p} \| |A^*|^{1-\alpha} C \|_{2q}. \quad (4.8)$$

In particular,

$$\|C^*Af(A)B\|_{2r} \leq B(f, \gamma; A) \| |A|^{1/2} B \|_{2p} \| |A^*|^{1/2} C \|_{2q} \quad (4.9)$$

for  $|A|^{1/2} B \in \mathcal{B}_{2p}(H)$  and  $|A^*|^{1/2} C \in \mathcal{B}_{2q}(H)$ .

*Proof.* Assume that  $\mathcal{E} = \{e_i\}_{i \in I}$  and  $\mathcal{F} = \{f_i\}_{i \in I}$  are orthonormal basis in  $H$ . Observe that we have  $\frac{1}{p} + \frac{1}{q} = 1$  and by Hölder's inequality for  $\frac{r}{p}$  and  $\frac{r}{q}$  we have

$$\begin{aligned} \sum_{i \in I} \langle |A|^\alpha B |e_i, e_i\rangle^r \langle |A^*|^{1-\alpha} C |f_i, f_i\rangle^r &= \sum_{i \in I} \left[ \langle |A|^\alpha B |e_i, e_i\rangle^p \right]^{\frac{r}{p}} \left[ \langle |A^*|^{1-\alpha} C |f_i, f_i\rangle^q \right]^{\frac{r}{q}} \\ &\leq \left( \sum_{i \in I} \langle |A|^\alpha B |e_i, e_i\rangle^p \right)^{r/p} \left( \sum_{i \in I} \langle |A^*|^{1-\alpha} C |f_i, f_i\rangle^q \right)^{r/q}. \end{aligned} \quad (4.10)$$

By McCarthy inequality for  $p, q > 1$  we get

$$\sum_{i \in I} \langle |A|^\alpha B |e_i, e_i\rangle^p \leq \sum_{i \in I} \langle |A|^\alpha B |^{2p} e_i, e_i\rangle$$

and

$$\sum_{i \in I} \langle |A^*|^{1-\alpha} C |f_i, f_i\rangle^q \leq \sum_{i \in I} \langle |A^*|^{1-\alpha} C |^{2q} f_i, f_i\rangle$$

and by (4.10)

$$\begin{aligned} \sum_{i \in I} \langle |A|^\alpha B |e_i, e_i\rangle^r \langle |A^*|^{1-\alpha} C |f_i, f_i\rangle^r &\leq \left( \sum_{i \in I} \langle |A|^\alpha B |^{2p} e_i, e_i\rangle \right)^{r/p} \left( \sum_{i \in I} \langle |A^*|^{1-\alpha} C |^{2q} f_i, f_i\rangle \right)^{r/q} \\ &= \| |A|^\alpha B \|_{2p}^{2r} \| |A^*|^{1-\alpha} C \|_{2q}^{2r}. \end{aligned} \quad (4.11)$$

By (4.3) and (4.11) we get

$$\sum_{i \in I} |\langle C^*Af(A)B e_i, f_i \rangle|^{2r} \leq B^{2r}(f, \gamma; A) \| |A|^\alpha B \|_{2p}^{2r} \| |A^*|^{1-\alpha} C \|_{2q}^{2r}. \quad (4.12)$$

Now, if we take the supremum over  $\mathcal{E}$  and  $\mathcal{F}$  in (4.12) we get

$$\|C^*Af(A)B\|_{2r}^{2r} \leq B^{2r}(f, \gamma; A) \| |A|^\alpha B \|_{2p}^{2r} \| |A^*|^{1-\alpha} C \|_{2q}^{2r}$$

and the inequality (4.8) is thus proved.  $\square$

**Remark 4.4.** If we take  $p = q = 2r = s \geq 1$ , then by (4.8) we get

$$\|C^*Af(A)B\|_s \leq B(f, \gamma; A) \| |A|^\alpha B \|_{2s} \| |A^*|^{1-\alpha} C \|_{2s} \tag{4.13}$$

provided that  $|A|^\alpha B \in \mathcal{B}_{2s}(H)$  and  $|A^*|^{1-\alpha} C \in \mathcal{B}_{2s}(H)$  for  $\alpha \in [0, 1]$ .  
For  $\alpha = 1/2$  we have

$$\|C^*Af(A)B\|_s \leq B(f, \gamma; A) \| |A|^{1/2} B \|_{2s} \| |A^*|^{1/2} C \|_{2s} \tag{4.14}$$

provided that  $|A|^{1/2} B \in \mathcal{B}_{2s}(H)$  and  $|A^*|^{1/2} C \in \mathcal{B}_{2s}(H)$ .  
If  $r = 2$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ , then

$$\|C^*Af(A)B\|_4 \leq B(f, \gamma; A) \| |A|^\alpha B \|_{2p} \| |A^*|^{1-\alpha} C \|_{2q} \tag{4.15}$$

provided that  $|A|^\alpha B \in \mathcal{B}_{2p}(H)$  and  $|A^*|^{1-\alpha} C \in \mathcal{B}_{2q}(H)$  for  $\alpha \in [0, 1]$ .  
In particular,

$$\|C^*Af(A)B\|_4 \leq B(f, \gamma; A) \| |A|^{1/2} B \|_{2p} \| |A^*|^{1/2} C \|_{2q} \tag{4.16}$$

for  $|A|^{1/2} B \in \mathcal{B}_{2p}(H)$  and  $|A^*|^{1/2} C \in \mathcal{B}_{2q}(H)$ .

**Theorem 4.5.** Let  $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$  be an analytic function on the domain  $G$  and  $A \in \mathcal{B}(H)$  with  $Sp(A) \subset G$  and  $\gamma$  a closed rectifiable path in  $G$  and such that  $Sp(A) \subset \text{ins}(\gamma)$ .

If  $r \geq 1/2$ ,  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $pr, qr \geq 1$  and  $|A|^\alpha B|^{2pr}, |A^*|^{1-\alpha} C|^{2qr} \in \mathcal{B}_1(H)$ , then  $C^*Af(A)B \in \mathcal{B}_{2r}(H)$  and

$$\omega_{2r}^{2r}(C^*Af(A)B) \leq B^{2r}(f, \gamma; A) \text{tr} \left( \frac{1}{p} |A|^\alpha B|^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C|^{2qr} \right). \tag{4.17}$$

If  $r \geq 1$  and  $|A|^\alpha B, |A^*|^{1-\alpha} C \in \mathcal{B}_{4r}(H)$ , then  $C^*Af(A)B \in \mathcal{B}_{2r}(H)$  and

$$\begin{aligned} \omega_{2r}^{2r}(C^*Af(A)B) &\leq \frac{1}{2} B^{2r}(f, \gamma; A) \left( \| |A|^\alpha B \|_{4r}^{2r} \| |A^*|^{1-\alpha} C \|_{4r}^{2r} + \omega_r^r \left( \| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right) \right) \\ &\leq \frac{1}{2} B^{2r}(f, \gamma; A) \left( \| |A|^\alpha B \|_{4r}^{2r} \| |A^*|^{1-\alpha} C \|_{4r}^{2r} + \| \| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \|_r^r \right). \end{aligned} \tag{4.18}$$

If  $r \geq 1$ ,  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $pr, qr \geq 2$  and  $|A|^\alpha B, |A^*|^{1-\alpha} C \in \mathcal{B}_1(H)$  then  $C^*Af(A)B \in \mathcal{B}_{2r}(H)$  and

$$\begin{aligned} \omega_{2r}^{2r}(C^*Af(A)B) &\leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[ \text{tr} \left( \frac{1}{p} |A|^\alpha B|^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C|^{2qr} \right) + \omega_r^r \left( \| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right) \right] \\ &\leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[ \text{tr} \left( \frac{1}{p} |A|^\alpha B|^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C|^{2qr} \right) + \| \| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \|_r^r \right]. \end{aligned} \tag{4.19}$$

*Proof.* From (2.7) for  $y = x$  we have that

$$|\langle C^*Af(A)Bx, x \rangle|^2 \leq B^2(f, \gamma; A) \langle |A|^\alpha B|^2 x, x \rangle \langle |A^*|^{1-\alpha} C|^2 x, x \rangle \tag{4.20}$$

for  $x \in H$  with  $\|x\| = 1$ .

If we take the power  $r > 0$ , we get, by Young and McCarthy inequalities, that

$$\begin{aligned} |\langle C^*Af(A)Bx, x \rangle|^{2r} &\leq B^{2r}(f, \gamma; A) \langle |A|^\alpha B|^2 x, x \rangle^r \langle |A^*|^{1-\alpha} C|^2 x, x \rangle^r \\ &\leq B^{2r}(f, \gamma; A) \left[ \frac{1}{p} \langle |A|^\alpha B|^2 x, x \rangle^{pr} + \frac{1}{q} \langle |A^*|^{1-\alpha} C|^2 x, x \rangle^{qr} \right] \\ &\leq B^{2r}(f, \gamma; A) \left[ \frac{1}{p} \langle |A|^\alpha B|^{2pr} x, x \rangle + \frac{1}{q} \langle |A^*|^{1-\alpha} C|^{2qr} x, x \rangle \right] \\ &= B^{2r}(f, \gamma; A) \left\langle \left( \frac{1}{p} |A|^\alpha B|^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C|^{2qr} \right) x, x \right\rangle \end{aligned}$$

for  $x \in H$  with  $\|x\| = 1$ .

If  $\mathcal{E} = \{e_i\}_{i \in I}$  is an orthonormal basis, then by taking  $x = e_i$  and summing over  $i \in I$  we get

$$\begin{aligned} \|C^*Af(A)B\|_{\mathcal{E}, 2r}^{2r} &= \sum_{i \in I} |\langle C^*Af(A)Be_i, e_i \rangle|^{2r} \\ &\leq B^{2r}(f, \gamma; A) \sum_{i \in I} \left\langle \left( \frac{1}{p} |A|^\alpha B|^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C|^{2qr} \right) e_i, e_i \right\rangle \\ &= B^{2r}(f, \gamma; A) \text{tr} \left( \frac{1}{p} |A|^\alpha B|^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C|^{2qr} \right), \end{aligned}$$

which, by taking the supremum over  $\mathcal{E}$ , proves (4.17).

By Buzano's inequality we have

$$\begin{aligned} \left\langle |A|^\alpha B|^2 x, x \right\rangle \left\langle x, |A^*|^{1-\alpha} C|^2 x \right\rangle &\leq \frac{1}{2} \left[ \left\| |A|^\alpha B|^2 x \right\| \left\| |A^*|^{1-\alpha} C|^2 x \right\| + \left| \left\langle |A|^\alpha B|^2 x, |A^*|^{1-\alpha} C|^2 x \right\rangle \right| \right] \\ &= \frac{1}{2} \left[ \left\| |A|^\alpha B|^2 x \right\| \left\| |A^*|^{1-\alpha} C|^2 x \right\| + \left| \left\langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 x, x \right\rangle \right| \right] \end{aligned}$$

for  $x \in H$  with  $\|x\| = 1$ .

If we take the power  $r \geq 1$  and use the convexity of power function, then we get

$$\begin{aligned} \left\langle |A|^\alpha B|^2 x, x \right\rangle^r \left\langle x, |A^*|^{1-\alpha} C|^2 x \right\rangle^r &\leq \left[ \frac{\left\| |A|^\alpha B|^2 x \right\| \left\| |A^*|^{1-\alpha} C|^2 x \right\| + \left| \left\langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 x, x \right\rangle \right|}{2} \right]^r \\ &\leq \frac{\left\| |A|^\alpha B|^2 x \right\|^r \left\| |A^*|^{1-\alpha} C|^2 x \right\|^r + \left| \left\langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 x, x \right\rangle \right|^r}{2} \\ &= \frac{\left\| |A|^\alpha B|^2 x \right\|^{2\frac{r}{2}} \left\| |A^*|^{1-\alpha} C|^2 x \right\|^{2\frac{r}{2}} + \left| \left\langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 x, x \right\rangle \right|^r}{2} \\ &= \frac{\left\langle |A|^\alpha B|^4 x, x \right\rangle^{\frac{r}{2}} \left\langle |A^*|^{1-\alpha} C|^4 x, x \right\rangle^{\frac{r}{2}} + \left| \left\langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 x, x \right\rangle \right|^r}{2} \end{aligned}$$

for  $x \in H$  with  $\|x\| = 1$ .

Therefore

$$\begin{aligned} \|C^* A f(A) B\|_{\mathcal{E}, 2r}^{2r} &= \sum_{i \in I} |C^* A f(A) B e_i, e_i|^{2r} \\ &\leq B^{2r}(f, \gamma; A) \sum_{i \in I} \left\langle |A|^\alpha B|^2 e_i, e_i \right\rangle^r \left\langle e_i, |A^*|^{1-\alpha} C|^2 e_i \right\rangle^r \\ &\leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[ \sum_{i \in I} \left\langle |A|^\alpha B|^4 e_i, e_i \right\rangle^{\frac{r}{2}} \left\langle |A^*|^{1-\alpha} C|^4 e_i, e_i \right\rangle^{\frac{r}{2}} + \sum_{i \in I} \left| \left\langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 e_i, e_i \right\rangle \right|^r \right]. \end{aligned} \tag{4.21}$$

Using Cauchy-Schwarz inequality we have

$$\begin{aligned} \sum_{i \in I} \left\langle |A|^\alpha B|^4 e_i, e_i \right\rangle^{\frac{r}{2}} \left\langle |A^*|^{1-\alpha} C|^4 e_i, e_i \right\rangle^{\frac{r}{2}} &\leq \left( \sum_{i \in I} \left\langle |A|^\alpha B|^4 e_i, e_i \right\rangle^r \right)^{1/2} \left( \sum_{i \in I} \left\langle |A^*|^{1-\alpha} C|^4 e_i, e_i \right\rangle^r \right)^{1/2} \\ &\leq \left( \sum_{i \in I} \left\langle |A|^\alpha B|^{4r} e_i, e_i \right\rangle \right)^{1/2} \left( \sum_{i \in I} \left\langle |A^*|^{1-\alpha} C|^{4r} e_i, e_i \right\rangle \right)^{1/2} \\ &= \| |A|^\alpha B \|_{4r}^{2r} \| |A^*|^{1-\alpha} C \|_{4r}^{2r}, \end{aligned}$$

where for the last inequality we used McCarthy's result for  $r \geq 1$ .

By taking the supremum over  $\mathcal{E}$ , we get the desired result (4.18).

Further, if we use Young's inequality for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q, \quad a, b \geq 0,$$

then we get

$$\begin{aligned} \left\| |A|^\alpha B|^2 x \right\|^r \left\| |A^*|^{1-\alpha} C|^2 x \right\|^r &\leq \frac{1}{p} \left\| |A|^\alpha B|^2 x \right\|^{pr} + \frac{1}{q} \left\| |A^*|^{1-\alpha} C|^2 x \right\|^{qr} \\ &= \frac{1}{p} \left\| |A|^\alpha B|^2 x \right\|^{2\frac{pr}{2}} + \frac{1}{q} \left\| |A^*|^{1-\alpha} C|^2 x \right\|^{2\frac{qr}{2}} \\ &= \frac{1}{p} \left\langle |A|^\alpha B|^4 x, x \right\rangle^{\frac{pr}{2}} + \frac{1}{q} \left\langle |A^*|^{1-\alpha} C|^4 x, x \right\rangle^{\frac{qr}{2}} \\ &\leq \frac{1}{p} \left\langle |A|^\alpha B|^{2pr} x, x \right\rangle + \frac{1}{q} \left\langle |A^*|^{1-\alpha} C|^{2qr} x, x \right\rangle \\ &= \left\langle \left( \frac{1}{p} |A|^\alpha B|^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C|^{2qr} \right) x, x \right\rangle \end{aligned}$$

for  $x \in H$  with  $\|x\| = 1$ .

Therefore

$$\begin{aligned} \|C^*Af(A)B\|_{\mathcal{E},2r}^{2r} &= \sum_{i \in I} |\langle C^*Af(A)Be_i, e_i \rangle|^{2r} \\ &\leq B^{2r}(f, \gamma; A) \sum_{i \in I} \left\langle |A|^\alpha B|^2 e_i, e_i \right\rangle^r \left\langle |A^*|^{1-\alpha} C|^2 e_i \right\rangle^r \\ &\leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[ \sum_{i \in I} \left\langle \left( \frac{1}{p} |A|^\alpha B|^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C|^{2qr} \right) e_i, e_i \right\rangle + \sum_{i \in I} \left\langle |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 e_i, e_i \right\rangle \right]^r \\ &= \frac{1}{2} B^{2r}(f, \gamma; A) \left[ \text{tr} \left( \frac{1}{p} |A|^\alpha B|^{2pr} + \frac{1}{q} |A^*|^{1-\alpha} C|^{2qr} \right) + \left\| |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 \right\|_{\mathcal{E},r}^r \right], \end{aligned}$$

which proves, by taking the supremum over  $\mathcal{E}$ , the desired inequality (4.19). □

**Remark 4.6.** Let  $\alpha \in [0, 1]$ . If  $r = 1/2$ ,  $p, q = 2$  and  $|A|^\alpha B|^2, |A^*|^{1-\alpha} C|^2 \in \mathcal{B}_1(H)$ , then  $C^*Af(A)B \in \mathcal{B}_1(H)$  and by (4.17) we get

$$\omega_1(C^*Af(A)B) \leq \frac{1}{2} B(f, \gamma; A) \text{tr} \left( |A|^\alpha B|^2 + |A^*|^{1-\alpha} C|^2 \right). \tag{4.22}$$

If  $r = 1$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then by (4.17) we obtain

$$\omega_2^2(C^*Af(A)B) \leq B^2(f, \gamma; A) \text{tr} \left( \frac{1}{p} |A|^\alpha B|^{2p} + \frac{1}{q} |A^*|^{1-\alpha} C|^{2q} \right), \tag{4.23}$$

provided that  $|A|^\alpha B|^{2p}, |A^*|^{1-\alpha} C|^{2q} \in \mathcal{B}_1(H)$ .

If we take  $r = 1$  in (4.18), then we get

$$\begin{aligned} \omega_2^2(C^*Af(A)B) &\leq \frac{1}{2} B^2(f, \gamma; A) \left( \| |A|^\alpha B \|^2_4 \| |A^*|^{1-\alpha} C \|^2_4 + \omega_1 \left( |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 \right) \right) \\ &\leq \frac{1}{2} B^2(f, \gamma; A) \left( \| |A|^\alpha B \|^2_4 \| |A^*|^{1-\alpha} C \|^2_4 + \left\| |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 \right\|_1 \right), \end{aligned} \tag{4.24}$$

provided that  $|A|^\alpha B, |A^*|^{1-\alpha} C \in \mathcal{B}_4(H)$ .

If  $r = 1$  and  $p = q = 2$  in (4.19), then we get for  $|A|^\alpha B|^{2p}, |A^*|^{1-\alpha} C|^{2q} \in \mathcal{B}_1(H)$  that

$$\begin{aligned} \omega_2^2(C^*Af(A)B) &\leq \frac{1}{4} B^2(f, \gamma; A) \left[ \text{tr} \left( |A|^\alpha B|^{2p} + |A^*|^{1-\alpha} C|^{2q} \right) + \frac{1}{2} B^2(f, \gamma; A) \omega_1 \left( |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 \right) \right] \\ &\leq \frac{1}{4} B^2(f, \gamma; A) \text{tr} \left( |A|^\alpha B|^{2p} + |A^*|^{1-\alpha} C|^{2q} \right) + \frac{1}{2} B^2(f, \gamma; A) \left\| |A^*|^{1-\alpha} C|^2 |A|^\alpha B|^2 \right\|_1. \end{aligned} \tag{4.25}$$

We also have:

**Theorem 4.7.** With the assumptions of Theorem 4.5, we have for  $r \geq 1, \lambda \in [0, 1]$  that

$$\omega_{2r}^{2r}(C^*Af(A)B) \leq B^{2r}(f, \gamma; A) \left\| (1-\lambda) |A|^\alpha B|^{2r} + \lambda |A^*|^{1-\alpha} C|^{2r} \right\| \times \| |A|^\alpha B \|_{2r}^{2r\lambda} \| |A^*|^{1-\alpha} C \|_{2r}^{2r(1-\lambda)}, \tag{4.26}$$

provided that  $|A|^\alpha B, |A^*|^{1-\alpha} C \in \mathcal{B}_{2r}(H)$ .

In particular,

$$\omega_{2r}^{2r}(C^*Af(A)B) \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left\| |A|^\alpha B|^{2r} + |A^*|^{1-\alpha} C|^{2r} \right\| \times \| |A|^\alpha B \|_{2r}^r \| |A^*|^{1-\alpha} C \|_{2r}^r. \tag{4.27}$$

*Proof.* If  $\mathcal{E} = \{e_i\}_{i \in I}$  is an orthonormal basis, then by taking  $x = e_i$  in (3.21) and summing over  $i \in I$  we get

$$\begin{aligned} \sum_{i \in I} |\langle C^*Af(A)Be_i, e_i \rangle|^{2r} &\leq B^{2r}(f, \gamma; A) \sum_{i \in I} \left[ \left\langle (1-\lambda) |A|^\alpha B|^{2r} + \lambda |A^*|^{1-\alpha} C|^{2r}, e_i, e_i \right\rangle \right] \\ &\quad \times \left\langle |A|^\alpha B|^2 e_i, e_i \right\rangle^{r\lambda} \left\langle |A^*|^{1-\alpha} C|^2 e_i, e_i \right\rangle^{r(1-\lambda)} \\ &\leq B^{2r}(f, \gamma; A) \left\| (1-\lambda) |A|^\alpha B|^{2r} + \lambda |A^*|^{1-\alpha} C|^{2r} \right\| \times \sum_{i \in I} \left\langle |A|^\alpha B|^2 e_i, e_i \right\rangle^{r\lambda} \left\langle |A^*|^{1-\alpha} C|^2 e_i, e_i \right\rangle^{r(1-\lambda)}. \end{aligned} \tag{4.28}$$

If we use Hölder's inequality for  $p = \frac{1}{\lambda}$ ,  $q = \frac{1}{1-\lambda}$ , then we have

$$\begin{aligned} \sum_{i \in I} \langle |A|^\alpha B|^2 e_i, e_i \rangle^{r\lambda} \langle |A^*|^{1-\alpha} C|^2 e_i, e_i \rangle^{r(1-\lambda)} &\leq \left( \sum_{i \in I} \langle |A|^\alpha B|^2 e_i, e_i \rangle^r \right)^\lambda \left( \sum_{i \in I} \langle |A^*|^{1-\alpha} C|^2 e_i, e_i \rangle^r \right)^{1-\lambda} \\ &\leq \left( \sum_{i \in I} \langle |A|^\alpha B|^{2r} e_i, e_i \rangle \right)^\lambda \left( \sum_{i \in I} \langle |A^*|^{1-\alpha} C|^{2r} e_i, e_i \rangle \right)^{1-\lambda} \\ &= \| |A|^\alpha B \|_{2r}^{2r\lambda} \| |A^*|^{1-\alpha} C \|_{2r}^{2r(1-\lambda)}. \end{aligned}$$

By taking the supremum over  $\mathcal{E}$ , we get the desired result (4.26).  $\square$

**Remark 4.8.** If we take  $r = 1$  in Theorem 4.7, then we get for  $\alpha \in [0, 1]$  that

$$\omega_2^2(C^* A f(A) B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) |A|^\alpha B|^2 + \lambda |A^*|^{1-\alpha} C|^2 \right\| \times \| |A|^\alpha B \|_2^{2\lambda} \| |A^*|^{1-\alpha} C \|_2^{2(1-\lambda)}, \quad (4.29)$$

provided that  $|A|^\alpha B, |A^*|^{1-\alpha} C \in \mathcal{B}_2(H)$ .

In particular,

$$\omega_2^2(C^* A f(A) B) \leq \frac{1}{2} B^2(f, \gamma; A) \left\| |A|^\alpha B|^2 + |A^*|^{1-\alpha} C|^2 \right\| \times \| |A|^\alpha B \|_2 \| |A^*|^{1-\alpha} C \|_2. \quad (4.30)$$

## 5. Some Examples

Consider the exponential function  $f(A) = \exp A$ ,  $A \in \mathcal{B}(H)$ . Assume that  $A \in \mathcal{B}(H)$  and  $\|A\| < R$  for some  $R > 0$ . Observe that for  $t \in [0, 1]$ ,

$$\left| \exp \left( R e^{2\pi i t} \right) \right| = \left| \exp [R (\cos(2\pi t) + i \sin(2\pi t))] \right| = \exp [R \cos(2\pi t)]$$

and then by (2.20) we get for  $B, C \in \mathcal{B}(H)$  that

$$\left| \langle C^* A \exp(A) B x, y \rangle \right| \leq \frac{R}{R - \|A\|} \int_0^1 \exp [R \cos(2\pi t)] dt \times \langle |A|^\alpha B|^2 x, x \rangle^{1/2} \langle |A^*|^{1-\alpha} C|^2 y, y \rangle^{1/2} \quad (5.1)$$

for  $x, y \in H$

The modified Bessel function of the first kind  $I_\nu(z)$  for real number  $\nu$  can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left( \frac{1}{2} z \right)^\nu \sum_{k=0}^{\infty} \frac{\left( \frac{1}{4} z^2 \right)^k}{k! \Gamma(\nu + k + 1)},$$

where  $\Gamma$  is the gamma function. For  $\nu = 0$  we have  $I_0(z)$  given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left( \frac{1}{4} z^2 \right)^k}{(k!)^2}.$$

An integral formula for real number  $\nu$  is

$$I_\nu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(\nu \theta) d\theta - \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-z \cosh t - \nu t} dt,$$

which simplifies for  $\nu$  an integer  $n$  to

$$I_n(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(n\theta) d\theta.$$

For  $n = 0$  we have

$$I_0(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} d\theta.$$

If we change the variable  $\theta = 2\pi t$ , then  $dt = \frac{1}{2\pi} d\theta$  and

$$\begin{aligned} \int_0^1 \exp [R \cos(2\pi t)] dt &= \frac{1}{2\pi} \int_0^{2\pi} \exp [R \cos \theta] d\theta \\ &= \frac{1}{2} \left( \frac{1}{\pi} \int_0^\pi \exp [R \cos \theta] d\theta + \frac{1}{\pi} \int_\pi^{2\pi} \exp [R \cos \theta] d\theta \right) \\ &= \frac{1}{2} (I_0(R) + I_0(-R)) = I_0(R). \end{aligned}$$

From (5.1) we then get

$$|\langle C^*A \exp(A)Bx, y \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \times \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1-\alpha} C \|^2 y, y \right\rangle^{1/2} \tag{5.2}$$

for  $\alpha \in [0, 1]$ ,  $x, y \in H$ ,  $A, B, C \in \mathcal{B}(H)$  with  $\|A\| < R$ .

By taking  $B = C = I$  in (5.2) we get for  $\|A\| < R$  that

$$|\langle A \exp(A)x, y \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \left\langle |A|^{2\alpha} x, x \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle^{1/2} \tag{5.3}$$

for  $x, y \in H$ . In particular,

$$|\langle A \exp(A)x, y \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \langle |A|x, x \rangle^{1/2} \langle |A^*|y, y \rangle^{1/2} \tag{5.4}$$

for  $x, y \in H$ .

If  $A$  is invertible and take  $C = I, B = A^{-1}$  in (5.2), then we get for  $\|A\| < R$  that

$$|\langle \exp(A)x, y \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \left\langle |A|^{-2(1-\alpha)} x, x \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle^{1/2} \tag{5.5}$$

for  $x, y \in H$ . In particular,

$$|\langle \exp(A)x, y \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \left\langle |A|^{-1} x, x \right\rangle^{1/2} \langle |A^*|y, y \rangle^{1/2} \tag{5.6}$$

for  $x, y \in H$ .

If  $0 < A$  with  $\|A\| < R$  and we take  $B = A^{-\beta}, C = A^{-1+\beta}, \alpha, \beta \in [0, 1]$ , then by (5.2) we derive

$$|\langle \exp(A)x, y \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \left\langle A^{2(\alpha-\beta)} x, x \right\rangle^{1/2} \left\langle A^{2(\beta-\alpha)} y, y \right\rangle^{1/2} \tag{5.7}$$

for  $x, y \in H$ .

By Theorem 3.1 we get the norm inequality

$$\|C^*A \exp(A)B\| \leq \frac{RI_0(R)}{R - \|A\|} \| |A|^\alpha B \| \| |A^*|^{1-\alpha} C \|. \tag{5.8}$$

We also have the numerical radius inequalities

$$\omega(C^*A \exp(A)B) \leq \frac{1}{2} \frac{RI_0(R)}{R - \|A\|} \left\| \| |A|^\alpha B \|^2 + \| |A^*|^{1-\alpha} C \|^2 \right\| \tag{5.9}$$

and

$$\omega^2(C^*A \exp(A)B) \leq \frac{1}{2} \left( \frac{RI_0(R)}{R - \|A\|} \right)^2 \times \left[ \| |A|^\alpha B \|^2 \| |A^*|^{1-\alpha} C \|^2 + \omega \left( \| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right) \right]. \tag{5.10}$$

Let  $r \geq 1/2, p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  and  $pr, qr \geq 1$ . If  $B, C \in \mathcal{B}(H)$  with  $|A|^\alpha B \in \mathcal{B}_{2pr}(H)$  and  $|A^*|^{1-\alpha} C \in \mathcal{B}_{2qr}(H)$  for  $\alpha \in [0, 1]$ , then  $C^*A \exp(A)B \in \mathcal{B}_{2r}(H)$  and by (4.1)

$$\|C^*A \exp(A)B\|_{2r} \leq \frac{RI_0(R)}{R - \|A\|} \| |A|^\alpha B \|_{2pr} \| |A^*|^{1-\alpha} C \|_{2qr}. \tag{5.11}$$

If  $r \geq 1/2, p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1, pr, qr \geq 1$  and  $\| |A|^\alpha B \|^{2pr}, \| |A^*|^{1-\alpha} C \|^{2qr} \in \mathcal{B}_1(H)$ , then  $C^*A \exp(A)B \in \mathcal{B}_{2r}(H)$  and by (4.17)

$$\begin{aligned} \omega_{2r}^{2r}(C^*A \exp(A)B) & \\ \leq \left( \frac{RI_0(R)}{R - \|A\|} \right)^{2r} \operatorname{tr} \left( \frac{1}{p} \| |A|^\alpha B \|^{2pr} + \frac{1}{q} \| |A^*|^{1-\alpha} C \|^{2qr} \right). & \end{aligned} \tag{5.12}$$

By using the power series

$$f(z) := \ln(1-z)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n} z^n$$

that is convergent on open disk  $D(0, 1)$ , we can define for all elements  $A$  in  $\mathcal{B}(H)$  with  $\|A\| < 1$ ,

$$\ln(I-A)^{-1} := \sum_{n=1}^{\infty} \frac{1}{n} A^n.$$

We observe that for  $|z| < 1$

$$\left| \ln(1-z)^{-1} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n} |z|^n = \ln(1-|z|)^{-1}.$$

Now if we assume that  $A, B, C \in \mathcal{B}(H)$  and  $\|A\| < R < 1$ , then by (2.22) we get

$$\left| \left\langle C^* A \ln(I-A)^{-1} Bx, y \right\rangle \right| \leq \frac{R \ln(1-R)^{-1}}{R - \|A\|} \times \left\langle \|A\|^\alpha B^2 x, x \right\rangle^{1/2} \left\langle \|A^*\|^{1-\alpha} C^2 y, y \right\rangle^{1/2} \quad (5.13)$$

for  $\alpha \in [0, 1]$ ,  $x, y \in H$ .

By taking  $B = C = I$  in (5.13) we get for  $\|A\| < R < 1$  that

$$\left| \left\langle A \ln(I-A)^{-1} x, y \right\rangle \right| \leq \frac{R \ln(1-R)^{-1}}{R - \|A\|} \left\langle |A|^{2\alpha} x, x \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle^{1/2} \quad (5.14)$$

for  $x, y \in H$ . In particular,

$$\left| \left\langle A \ln(I-A)^{-1} x, y \right\rangle \right| \leq \frac{R \ln(1-R)^{-1}}{R - \|A\|} \langle |A| x, x \rangle^{1/2} \langle |A^*| y, y \rangle^{1/2} \quad (5.15)$$

for  $x, y \in H$ .

If  $A$  is invertible and take  $C = I, B = A^{-1}$  in (5.13), then we get for  $\|A\| < R < 1$  that

$$\left| \left\langle \ln(I-A)^{-1} x, y \right\rangle \right| \leq \frac{R \ln(1-R)^{-1}}{R - \|A\|} \left\langle |A|^{-2(1-\alpha)} x, x \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle^{1/2} \quad (5.16)$$

for  $x, y \in H$ . In particular,

$$\left| \left\langle \ln(I-A)^{-1} x, y \right\rangle \right| \leq \frac{R \ln(1-R)^{-1}}{R - \|A\|} \left\langle |A|^{-1} x, x \right\rangle^{1/2} \langle |A^*| y, y \rangle^{1/2}. \quad (5.17)$$

If  $0 < A$  with  $\|A\| < R < 1$  and we take  $B = A^{-\beta}, C = A^{-1+\beta}, \beta \in [0, 1]$  in (5.13), then we derive

$$\left| \left\langle \ln(I-A)^{-1} x, y \right\rangle \right| \leq \frac{R \ln(1-R)^{-1}}{R - \|A\|} \left\langle A^{2(\alpha-\beta)} x, x \right\rangle^{1/2} \left\langle A^{2(\beta-\alpha)} y, y \right\rangle^{1/2} \quad (5.18)$$

for  $\alpha \in [0, 1]$ ,  $x, y \in H$ .

One can state some norm, numerical radius and  $p$ -Schatten norm inequalities for  $A \ln(I-A)^{-1}$ , however the details are omitted.

## Article Information

**Acknowledgements:** The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

**Author's contributions:** All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

**Conflict of Interest Disclosure:** No potential conflict of interest was declared by the author.

**Copyright Statement:** Authors own the copyright of their work published in the journal and their work is published under the CC BY-NC 4.0 license.

**Supporting/Supporting Organizations:** No grants were received from any public, private or non-profit organizations for this research.

**Ethical Approval and Participant Consent:** It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

**Plagiarism Statement:** This article was scanned by the plagiarism program. No plagiarism detected.

**Availability of data and materials:** Not applicable.

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