Power vector inequalities for operator pairs in Hilbert spaces and their applications

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Research Article

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Abstract: This study explores the power vector inequalities for a pair of operators (B, C) in a Hilbert space. By utilizing a Mitrinović-Pečarić-Fink-type inequality for inner products and norms, we derive various power vector inequalities. Specifically, we consider the cases where (B, C) is equal to (A, A^*) or $(\mathrm{Re}(A), \mathrm{Im}(A))$ for an operator A in $B(H)$, where H is a Hilbert space. This leads to the derivation of vector, norm, and numerical radius inequalities for a single operator. Furthermore, we obtain power inequalities for the s-r-norm and s-r -numerical radius of the operator pair $(B, C) \in B(H)$, which generalizes the Euclidean norm and Euclidean numerical radius. Finally, we apply these results to derive the corresponding inequalities for a single operator $A \in B(H)$.

Keywords: power vector inequalities, operators, Hilbert space, Mitrinović-Pečarić-Fink inequality, norm inequalities

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1 Introduction

Let H be a complex Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and a corresponding norm $\|\cdot\|$. We denote the C^*-algebra of bounded linear operators on H as $B(H)$. For any operator $A \in B(H)$, the adjoint of A is denoted as A^*, and $|A| = (A^*A)^{\frac{1}{2}}$ represents the positive square root of A. The real and imaginary parts of A are defined as $\mathrm{Re}(A) = \frac{1}{2}(A + A^*)$ and $\mathrm{Im}(A) = \frac{1}{2\mathrm{i}}(A - A^*)$, respectively. We define the numerical range of A, denoted by $W(A)$, as the set of values $\{\langle Ax, x \rangle : x \in H, \|x\| = 1\}$.

Let $\|A\|$ and $w(A)$ denote the operator norm and the numerical radius of A, respectively. The operator norm is given by

$$\|A\| = \sup\{|\langle Ax, y \rangle|;\ x, y \in H, \|x\| = \|y\| = 1\},$$

while the numerical radius is defined as

$$w(A) = \sup\{|\langle Ax, x \rangle|;\ x \in H, \|x\| = 1\}.$$

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It is known that the numerical radius $w(\cdot)$ defines a norm on $B(H)$ and is equivalent to the operator norm $\|\cdot\|$. In fact, the following double inequality holds:

$$\frac{1}{2}\|A\| \le w(A) \le \|A\|. \tag{1.1}$$

These inequalities are sharp. The first inequality becomes an equality if $A^2 = 0$, while the second inequality becomes an equality if A is a normal operator. An improvement to these inequalities was established by Kittaneh [1], who showed that

$$\frac{1}{2}\sqrt{\|A^*A + AA^*\|} \le w(A) \le \frac{\sqrt{2}}{2}\sqrt{\|A^*A + AA^*\|}. \tag{1.2}$$

For further advancements in (1.1) and (1.2), interested readers can refer to [2–9].

For an operator $T \in B(H)$, let $r(T)$ denote the spectral radius of T. It is well known that for every $T \in B(H)$, we have the fundamental inequality

$$r(T) \le w(T) \tag{1.3}$$

and that equality holds in inequality (1.3) if T is normal.

In addition to inequality (1.3), the most important properties of the spectral radius are the spectral radius formula

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}, \tag{1.4}$$

a special case of the spectral mapping theorem, which asserts that

$$r(T^m) = r^m(T) \tag{1.5}$$

for every natural number m, and the commutativity property, which asserts that

$$r(AB) = r(BA) \quad \text{for every } A, B \in B(H). \tag{1.6}$$

It follows from the spectral radius formula (1.4) that if $A, B \in B(H)$ are commutative, then the following subadditivity

$$r(A + B) \le r(A) + r(B) \tag{1.7}$$

and submultiplicativity

$$r(AB) \le r(A)r(B) \tag{1.8}$$

properties hold. We also observe that any upper bound for the numerical radius will provide an upper bound for the spectral radius, which may motivate finding upper bounds for the latter in order to estimate the former. For additional properties of the spectral radius, the reader is referred to the classical book [10].

Let B and C be operators in $B(H)$. The Euclidean operator radius between B and C, denoted as $w_e(B, C)$, is defined as

$$w_e(B, C) = \sup\{\sqrt{|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2}\,;\ x \in H, \|x\| = 1\}.$$

More information can be found in [11–13].

According to [13], the function $w_e(\cdot,\cdot) : B(H) \times B(H) \to [0, \infty)$ is a norm that satisfies the inequality

$$\frac{\sqrt{2}}{4}\sqrt{\||B|^2 + |C|^2\|} \le w_e(B, C) \le \sqrt{\||B|^2 + |C|^2\|}. \tag{1.9}$$

The constants $\frac{\sqrt{2}}{4}$ and 1 are the optimal choices in (1.9). If B and C are self-adjoint operators, then (1.9) simplifies to

$$\frac{\sqrt{2}}{4}\sqrt{\|B^2 + C^2\|} \le w_e(B, C) \le \sqrt{\|B^2 + C^2\|}.$$

Note that for self-adjoint operators B and C, $w_e(B, C) = w(B + iC)$. The proof of this follows easily from the definition of $w_e(B, C)$.

In a previous study [14], the second author derived a lower bound as follows:

$$\frac{\sqrt{2}}{2}[w(B^2+C^2)]^{\frac{1}{2}} \le w_e(B,C). \tag{1.10}$$

It was shown that the constant $\frac{\sqrt{2}}{2}$ in (1.10) cannot be replaced by a larger constant. The same study also obtained the following results:

$$\frac{\sqrt{2}}{2}\max\{w(B+C), w(B-C)\} \le w_e(B,C) \le \frac{\sqrt{2}}{2}\sqrt{w^2(B+C)+w^2(B-C)},$$

where the constant $\frac{\sqrt{2}}{2}$ is sharp in both inequalities. Furthermore, the study presented the following inequality:

$$w_e^2(B,C) \le \max\{\|B\|^2, \|C\|^2\} + w(C^*B).$$

which is also sharp. Another sharp inequality derived in the study is as follows:

$$w_e^2(B,C) \le \frac{1}{2}[\|B^*B+C^*C\| + \|B^*B-C^*C\|] + w(C^*B).$$

By considering $(B,C)=(A,A^*)$ or $(B,C)=(\mathrm{Re}(A), \mathrm{Im}(A))$ for an operator $A \in B(H)$, the second author derived several norm and numerical radius inequalities for a single operator A in [14].

For additional refinements of the aforementioned results, readers are advised to refer to the recent study by Jana et al. [15]. In the same study, they also presented the following interesting results as well:

$$w_e^2(B,C) \le \min\{w^2(B+C), w^2(B-C)\} + \frac{1}{2}\||C|^2+|B^*|^2\| + w(BC),$$

$$\max_{0\le\alpha\le1} w(\alpha B^2+(1-\alpha)C^2) \le w_e^2(B,C),$$

and

$$w_e^2(B,C) \le w^2(\sqrt{\alpha}B+\sqrt{1-\alpha}C) + w^2(\sqrt{1-\alpha}B+\sqrt{\alpha}C)$$

for all $\alpha \in [0,1]$.

In [16], for $r \ge 1$, the generalization of the numerical radius for a pair of operators $B, C \in B(H)$, referred to as the generalized s-r-numerical radius, is defined by

$$w_r(B,C) \coloneqq \sup_{\|x\|=1}(|\langle x,Bx\rangle|^r + |\langle x,Cx\rangle|^r)^{\frac{1}{r}}$$

and the generalized s-r-norm is defined by

$$\|(B,C)\|_r \coloneqq \sup_{\|x\|=\|y\|=1}(|\langle x,By\rangle|^r + |\langle x,Cy\rangle|^r)^{\frac{1}{r}}.$$

If we choose $r=2$ in the previous definitions, then we obtain the Euclidean numerical radius

$$w_e(B,C) \coloneqq \sup_{\|x\|=1}(|\langle x,Bx\rangle|^2 + |\langle x,Cx\rangle|^2)^{\frac{1}{2}}$$

and the Euclidean norm

$$\|(B,C)\|_e \coloneqq \sup_{\|x\|=\|y\|=1}(|\langle x,By\rangle|^2 + |\langle x,Cy\rangle|^2)^{\frac{1}{2}}$$

studied in [13].

For $r=1$, we denote

$$w(B,C) \coloneqq \sup_{\|x\|=1}(|\langle x,Bx\rangle| + |\langle x,Cx\rangle|)$$

and

$$\|(B,C)\| \coloneqq \sup_{\|x\|=\|y\|=1}(|\langle x,By\rangle| + |\langle x,Cy\rangle|).$$

The above notations can also be extended for $r \in (0,1)$; however in this case, they are no longer norms.

In 2017, Moslehian et al. [11] obtained the following fundamental inequalities for the *s-r-numerical radius* of the pair of operators $B, C \in B(H)$:

$$w_p(B,C) \leq w_q(B,C) \leq 2^{\frac{1}{q}-\frac{1}{p}} w_p(B,C), \quad \text{for } p \geq q \geq 1,$$

$$2^{\frac{1}{p}-2}\|BB^* + CC^*\| \leq w_p(B,C), \qquad p \geq 2,$$

$$2^{\frac{1}{p}-1}\max\{w(B+C), w(B-C)\} \leq w_p(B,C),$$

$$2^{\frac{1}{p}-1}\max\{w(B), w(C)\} \leq w_p(B,C), \qquad p \geq 1,$$

and

$$2^{\frac{1}{p}-1} w^{\frac{1}{2}}(B^2 + C^2) \leq w_p(B,C), \quad p \geq 1$$

to mention a few. Moslehian et al. [11] also applied their results for the Cartesian decomposition of an operator A.

We conclude this introductory section by recalling a recent generalization of the Boas-Bellman result (see [17,18] or [19, p. 392]) given by Mitrinović et al. [19, p. 392]. They proved that if $x, y_1, \ldots, y_n$ are elements of H (which need only to be an inner product space, not necessarily complete), the following inequality holds:

$$\left|\sum_{i=1}^{n} c_i \langle x, y_i\rangle\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left[\max_{1\leq i\leq n} \|y_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle y_i, y_j\rangle|^2\right)^{\frac{1}{2}}\right]. \tag{1.11}$$

It should be noted that if we choose $c_i = \overline{\langle x, y_i\rangle}$ in (1.11), we obtain the well-known inequality in [19]. For further results regarding the Boas-Bellman-type inequality, Mitrinović-Pečarić-Fink-type inequality, and Bessel inequality, we invite readers to consult [19–22] and references therein.

Motivated by the results in [15], in this work, we first employ the Mitrinović-Pečarić-Fink-type inequality (1.11) to derive several power vector inequalities for a pair of operators (B, C) in Section 2.

The main result in this section that plays a crucial role in the other sections is the following power inequality for $B, C \in B(H)$ and $\alpha, \beta \in \mathbb{C}$:

$$|\langle x, (\alpha B + \beta C)y\rangle|^{2p} \leq 2^{3\frac{p}{2}-1}\, \|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p |\langle C^*By, y\rangle|^p + 2^{p-1}\, \|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p \max\{\|By\|^{2p}, \|Cy\|^{2p}\}$$

for all $x, y \in H$ and $p \geq 1$.

In particular, for $p = 1$, we establish that for $B, C \in B(H)$, $\alpha, \beta \in \mathbb{C}$ and $x, y \in H$, we have the following *sharp inequality*

$$|\langle x, (\alpha B + \beta C)y\rangle|^2 \leq \sqrt{2}\, \|x\|^2(|\alpha|^2 + |\beta|^2)|\langle C^*By, y\rangle| + \|x\|^2(|\alpha|^2 + |\beta|^2)\max\{\|By\|^2, \|Cy\|^2\},$$

which, for an appropriate choice of the operators, reduces to Bessel's inequality. By considering $(B, C) = (A, A^*)$ or $(B, C) = (\operatorname{Re}(A), \operatorname{Im}(A))$ for $A \in B(H)$, we obtain vector, norm, and numerical radius inequalities for a single operator.

In Section 3, motivated by the results of Moslehian et al. in [11], we explore several power inequalities for the s-r-norm and s-r-numerical radius of the pair of operators $(B, C) \in B(H)$, which generalize the Euclidean norm and Euclidean numerical radius.

Two of the main results we want to emphasize here are

$$\|(B,C)\|_r^{2pr} \leq 2^{p-1}\, \|(B,C)\|_{2(r-1)}^{2p(r-1)}\left[2^{\frac{p}{2}} w^p(C^*B) + \max\{\|B\|^{2p}, \|C\|^{2p}\}\right]$$

and

$$w_r^{2pr}(B,C) \leq 2^{p-1} w_{2(r-1)}^{2p(r-1)}(B,C)\left[2^{\frac{p}{2}} w^p(C^*B) + \max\{\|B\|^{2p}, \|C\|^{2p}\}\right]$$

for $B, C \in B(H)$, $p \geq 1$, and $r > 1$. They provide different generalizations for powers and more diverse upper bounds than those from [11,14,15] mentioned above. Due to their quite different analytic expressions, they cannot be compared in general with the bounds listed above. Finally, we apply these results to derive the some power inequalities for a single operator $A \in B(H)$.

2 Vector inequalities

The main objective of this section is to derive power vector inequalities for a pair of operators (B, C). To establish our first result in this direction, we will utilize the Mitrinović-Pečarić-Fink-type inequality (1.11) for $n = 2$. Specifically, for $n = 2$ in (1.11), we obtain

$$|c_1\langle x, y_1\rangle + c_2\langle x, y_2\rangle|^2 \leq \|x\|^2(|c_1|^2 + |c_2|^2)[\max\{\|y_1\|^2, \|y_2\|^2\} + \sqrt{2}|\langle y_1, y_2\rangle|] \tag{2.1}$$

for complex numbers c_1, c_2 and vectors $x, y_1, y_2 \in H$.

We are now prepared to state the following result.

Theorem 2.1. *Let $B, C \in B(H)$ and $\alpha, \beta \in \mathbb{C}$. Then, for all $x, y \in H$ and $p \geq 1$ we have*:

$$\begin{aligned} &|\langle x, (\alpha B + \beta C)y\rangle|^{2p} \\ &\leq 2^{3\frac{p}{2}-1}\|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p|\langle C^*By, y\rangle|^p + 2^{p-1}\|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p\max\{\|By\|^{2p}, \|Cy\|^{2p}\} \\ &\leq 2^{3\frac{p}{2}-1}\|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p|\langle C^*By, y\rangle|^p + 2^{p-2}\|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p \\ &\quad \times [\langle(|B|^2 + |C|^2)y, y\rangle^p + |\langle(|B|^2 - |C|^2)y, y\rangle|^p]. \end{aligned} \tag{2.2}$$

Proof. Note first that for all $x, y \in H$, we have

$$\begin{aligned} \max\{\|By\|^2, \|Cy\|^2\} &= \max\{\langle|B|^2y, y\rangle, \langle|C|^2y, y\rangle\} \\ &= \frac{1}{2}[\langle|B|^2y, y\rangle + \langle|C|^2y, y\rangle] + \frac{1}{2}|\langle|B|^2y, y\rangle - \langle|C|^2y, y\rangle| \\ &= \frac{1}{2}\langle(|B|^2 + |C|^2)y, y\rangle + \frac{1}{2}|\langle(|B|^2 - |C|^2)y, y\rangle|. \end{aligned} \tag{2.3}$$

Let now $x, y \in H$ and $\alpha, \beta \in \mathbb{C}$. If we take in (2.1) $c_1 = \bar{\alpha}$, $c_2 = \bar{\beta}$, $y_1 = By$, and $y_2 = Cy$ and then, using (2.3), we obtain

$$|\langle x, (\alpha B + \beta C)y\rangle|^2 \leq \|x\|^2(|\alpha|^2 + |\beta|^2)[\max\{\|By\|^2, \|Cy\|^2\} + \sqrt{2}|\langle C^*By, y\rangle|].$$

If we take the power $p \geq 1$, then we obtain

$$|\langle x, (\alpha B + \beta C)y\rangle|^{2p} \leq \|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p[\max\{\|By\|^2, \|Cy\|^2\} + \sqrt{2}|\langle C^*By, y\rangle|]^p \tag{2.4}$$

for all x, $y \in H$.

Using the elementary inequality that follows from the convexity of the power function,

$$(m + n)^p \leq 2^{p-1}(m^p + n^p), \quad m, n \geq 0,\ p \geq 1,$$

we obtain

$$[\max\{\|By\|^2, \|Cy\|^2\} + \sqrt{2}|\langle C^*By, y\rangle|]^p \leq 2^{p-1}\left[\max\{\|By\|^{2p}, \|Cy\|^{2p}\} + 2^{\frac{p}{2}}|\langle C^*By, y\rangle|^p\right],$$

which proves the first inequality in (2.2).

Now, observe that

$$\begin{aligned} \max\{\|By\|^{2p}, \|Cy\|^{2p}\} &= (\max\{\|By\|^2, \|Cy\|^2\})^p \\ &= \left[\frac{\langle(|B|^2 + |C|^2)y, y\rangle + |\langle(|B|^2 - |C|^2)y, y\rangle|}{2}\right]^p \leq \frac{\langle(|B|^2 + |C|^2)y, y\rangle^p + |\langle(|B|^2 - |C|^2)y, y\rangle|^p}{2} \end{aligned}$$

for all x, $y \in H$, which proves the last part of (2.2). □

The next corollary presents a specific instance of the aforementioned outcome.

Corollary 2.1. *Let* $B, C \in B(H)$ *and* $\alpha, \beta \in \mathbb{C}$. *Then, the following inequality*

$$|\langle x, (\alpha B + \beta C)y\rangle|^2 \leq \sqrt{2}\ \|x\|^2(|\alpha|^2 + |\beta|^2)|\langle C^*By, y\rangle| + \|x\|^2(|\alpha|^2 + |\beta|^2)\max\{\|By\|^2, \|Cy\|^2\} \tag{2.5}$$

holds for all $x, y \in H$. *Moreover, this inequality is sharp.*

Proof. The result follows immediately by setting $p = 1$ in Theorem 2.1. Now, we prove that inequality (2.5) is sharp.

Let $e, f \in H$ such that $e \perp f$ and $\|e\| = \|f\| = 1$. Take $y \in H$ and two operators B and C such that $By = e$ and $Cy = f$. For instance, we can consider the projections $Bu := \langle u, e\rangle e$ and $Cu := \langle u, f\rangle f$, $u \in H$. If we choose $y = e + f$, then

$$B(y) := \langle e + f, e\rangle e = e \quad \text{and} \quad C(y) := \langle e + f, f\rangle f = f.$$

Now, if we replace these in (2.5), then we obtain

$$|\alpha\langle x, e\rangle + \beta\langle x, f\rangle|^2 \leq \|x\|^2(|\alpha|^2 + |\beta|^2) \tag{2.6}$$

that holds for any $\alpha, \beta \in C$ and $x \in H$.

Let $\alpha = \overline{\langle x, e\rangle}$ and $\beta = \overline{\langle x, f\rangle}$, then by (2.6) we derive

$$(|\langle x, e\rangle|^2 + |\langle x, f\rangle|^2)^2 \leq \|x\|^2(|\langle x, e\rangle|^2 + |\langle x, f\rangle|^2),$$

namely, Bessel's inequality

$$|\langle x, e\rangle|^2 + |\langle x, f\rangle|^2 \leq \|x\|^2, \quad \text{for any } x \in H,$$

which is a nontrivial sharp inequality. □

Theorem 2.1 leads to several important results and practical uses. One significant outcome is described in the next corollary.

Corollary 2.2. *Let* $B, C \in B(H)$, $\alpha, \beta \in \mathbb{C}$, *and* $p \geq 1$. *Then, we have*

$$\begin{aligned} \|\alpha B + \beta C\|^{2p} &\leq 2^{3\frac{p}{2}-1}(|\alpha|^2 + |\beta|^2)^p w^p(C^*B) + 2^{p-1}(|\alpha|^2 + |\beta|^2)^p \max\{\|B\|^{2p}, \|C\|^{2p}\} \\ &\leq 2^{3\frac{p}{2}-1}(|\alpha|^2 + |\beta|^2)^p w^p(C^*B) + 2^{p-2}(|\alpha|^2 + |\beta|^2)^p[\||B|^2 + |C|^2\|^p + \||B|^2 - |C|^2\|^p]. \end{aligned} \tag{2.7}$$

Proof. If we take in (2.2), the supremum over $x \in H$ with $\|x\| = 1$, then we obtain the vector norm inequality for $B, C \in B(H)$, $\alpha, \beta \in \mathbb{C}$ and $p \geq 1$

$$\begin{aligned} \|(\alpha B + \beta C)y\|^{2p} &\leq 2^{3\frac{p}{2}-1}(|\alpha|^2 + |\beta|^2)^p|\langle C^*By, y\rangle|^p + 2^{p-1}(|\alpha|^2 + |\beta|^2)^p \max\{\|By\|^{2p}, \|Cy\|^{2p}\} \\ &\leq 2^{3\frac{p}{2}-1}(|\alpha|^2 + |\beta|^2)^p|\langle C^*By, y\rangle|^p + 2^{p-2}(|\alpha|^2 + |\beta|^2)^p \\ &\quad \times [\langle(|B|^2 + |C|^2)y, y\rangle^p + |\langle(|B|^2 - |C|^2)y, y\rangle|^p] \end{aligned} \tag{2.8}$$

for all $y \in H$.

If we take the supremum over $y \in H$ with $\|y\| = 1$ in (2.8), then we obtain the desired result. □

Some other notable consequences of Theorem 2.1 are summarized in the following remark.

Remark 2.1. (1) Given $A \in B(H)$, if we take $B = A$ and $C = A^*$ in (2.2), we obtain

$$\begin{aligned} &|\langle x, (\alpha A + \beta A^*)y\rangle|^{2p} \\ &\leq\ 2^{3\frac{p}{2}-1}\ \|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p|\langle A^2y, y\rangle|^p + 2^{p-1}\ \|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p \max\{\|Ay\|^{2p}, \|A^*y\|^{2p}\}, \\ &\leq 2^{3\frac{p}{2}-1}\ \|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p|\langle A^2y, y\rangle|^p + 2^{p-2}\ \|x\|^{2p}(|\alpha|^2 + |\beta|^2)^p \\ &\quad \times [\langle(|A|^2 + |A^*|^2)y, y\rangle^p + |\langle(|A|^2 - |A^*|^2)y, y\rangle|^p] \end{aligned}$$

for all x, $y \in H$. In the case $p = 1$, we obtain

$$|\langle x, (\alpha A + \beta A^*)y\rangle|^2 \leq \sqrt{2}\ \|x\|^2(|\alpha|^2 + |\beta|^2)|\langle A^2y, y\rangle| + \|x\|^2(|\alpha|^2 + |\beta|^2)\max\{\|Ay\|^2, \|A^*y\|^2\}$$

for all $x,\ y \in H$.

For $\alpha = \beta = \frac{1}{2}$, we have

$$\begin{aligned}|\langle x, \operatorname{Re}(A)y\rangle|^{2p} &\leq 2^{\frac{p}{2}-1}\|x\|^{2p}\,|\langle A^2y, y\rangle|^p + \frac{1}{2}\max\{\|Ay\|^{2p}, \|A^*y\|^{2p}\}\\ &\leq 2^{\frac{p}{2}-1}\|x\|^{2p}\,|\langle A^2y, y\rangle|^p + \frac{1}{4}[\langle(|A|^2 + |A^*|^2)y, y\rangle^p + |\langle(|A|^2 - |A^*|^2)y, y\rangle|^p],\end{aligned}$$

while for $\alpha = \frac{1}{2i}$, $\beta = -\frac{1}{2i}$, we obtain

$$\begin{aligned}&|\langle x, \operatorname{Im}(A)y\rangle|^{2p}\\ &\quad\leq\ 2^{\frac{p}{2}-1}\|x\|^{2p}\,|\langle A^2y, y\rangle|^p + \frac{1}{2}\ \|x\|^{2p}\max\{\|Ay\|^{2p}, \|A^*y\|^{2p}\}\\ &\quad\leq\ 2^{\frac{p}{2}-1}\|x\|^{2p}\,|\langle A^2y, y\rangle|^p + \frac{1}{4}\ \|x\|^{2p}[\langle(|A|^2 + |A^*|^2)y, y\rangle^p + |\langle(|A|^2 - |A^*|^2)y, y\rangle|^p]\end{aligned}$$

for all $x,\ y \in H$. Thus, we deduce the following norm inequalities:

$$\begin{aligned}&\max\{\|\operatorname{Re}(A)\|^{2p}, \|\operatorname{Im}(A)\|^{2p}\}\\ &\quad\leq\ 2^{\frac{p}{2}-1}w^p(A^2) + \frac{1}{2}\ \|A\|^{2p}\\ &\quad\leq\ 2^{\frac{p}{2}-1}w^p(A^2) + \frac{1}{4}[\||A|^2 + |A^*|^2\|^p + \||A|^2 - |A^*|^2\|^p].\end{aligned}$$

(2) Let $A \in B(H)$ and $A = \operatorname{Re}(A) + i\operatorname{Im}(A)$ be its Cartesian decomposition, then by taking $B = \operatorname{Re}(A)$, $C = \operatorname{Im}(A)$, $\alpha = 1$, and $\beta = i$ in (2.2), we obtain

$$\begin{aligned}|\langle x, Ay\rangle|^{2p} &\leq 2^{5\frac{p}{2}-1}\ \|x\|^{2p}|\langle\operatorname{Im}(A)\operatorname{Re}(A)y, y\rangle|^p + 2^{2p-1}\ \|x\|^{2p}\max\{\|\operatorname{Re}(A)y\|^{2p}, \|\operatorname{Im}(A)y\|^{2p}\}\\ &\leq 2^{5\frac{p}{2}-1}\ \|x\|^{2p}|\langle\operatorname{Im}(A)\operatorname{Re}(A)y, y\rangle|^p + 2^{2p-2}\ \|x\|^{2p}\\ &\quad\times[\langle(\operatorname{Re}^2(A) + \operatorname{Im}^2(A))y, y\rangle^p + |\langle(\operatorname{Re}^2(A) - \operatorname{Im}^2(A))y, y\rangle|^p]\end{aligned}$$

for all $x,\ y \in H$. We immediately deduce the following norm inequalities:

$$\begin{aligned}\|A\|^{2p} &\leq 2^{5\frac{p}{2}-1}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + 2^{2p-1}\max\{\|\operatorname{Re}(A)\|^{2p}, \|\operatorname{Im}(A)\|^{2p}\}\\ &\leq 2^{5\frac{p}{2}-1}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + 2^{2p-2}[\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\|^p + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|^p].\end{aligned}$$

For the Cartesian decomposition of an operator, we can state the following result for the numerical radius as well.

Proposition 2.1. *For all $A \in B(H)$, we have the numerical radius inequality*

$$\begin{aligned}w^{2p}(A) &\leq 2^{3\frac{p}{2}-1}w^p(\operatorname{Im}(A)\operatorname{Re}A) + 2^{p-1}\max\{\|\operatorname{Im}(A)\|^{2p}, \|\operatorname{Re}(A)\|^{2p}\}\\ &\leq 2^{3\frac{p}{2}-1}w^p(\operatorname{Im}(A)\operatorname{Re}A) + 2^{p-2}[\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\|^p + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|^p]\end{aligned} \tag{2.9}$$

for all $p \geq 1$.

Proof. We use the representation of the numerical radius, see for instance [23, Theorem 2.2.11]

$$w(A) = \sup_{\theta\in\mathbb{R}}\|\operatorname{Re}(e^{i\theta}A)\|.$$

Observe that

$$\|\operatorname{Re}(e^{i\theta}A)\| = \|\operatorname{Re}((\cos\theta + i\sin\theta)(\operatorname{Re}(A) + i\operatorname{Im}(A)))\| = \|\cos\theta\operatorname{Re}(A) - \sin\theta\operatorname{Im}(A)\|$$

for all $\theta \in \mathbb{R}$.

From (2.7), for $\alpha = \cos\theta$, $\beta = -\sin\theta$, $B = \operatorname{Re}(A)$, and $C = \operatorname{Im}(A)$, we obtain

$$\begin{aligned}\|\operatorname{Re}(e^{i\theta}A)\|^{2p} &\le 2^{3\frac{p}{2}-1}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + 2^{p-1}\max\{\|\operatorname{Im}(A)\|^{2p}, \|\operatorname{Re}(A)\|^{2p}\}\\ &\le 2^{3\frac{p}{2}-1}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + 2^{p-2}[\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\|^p + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|^p]\end{aligned}$$

and by taking the supremum over $\theta \in \mathbb{R}$, we obtain (2.9). □

Remark 2.2. For $p = 1$, we obtain

$$\begin{aligned}w^2(A) &\le \sqrt{2}\,w(\operatorname{Im}(A)\operatorname{Re}(A)) + \max\{\|\operatorname{Im}(A)\|^2, \|\operatorname{Re}(A)\|^2\}\\ &\le \sqrt{2}\,w(\operatorname{Im}(A)\operatorname{Re}(A)) + \frac{1}{2}[\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\| + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|].\end{aligned} \tag{2.10}$$

Let us consider now

$$H_\theta \coloneqq \operatorname{Re}(e^{i\theta}A) = \frac{1}{2}(e^{i\theta}A + e^{-i\theta}A^*).$$

According to the proof of the previous result, we have

$$H_\theta = \cos\theta\operatorname{Re}(A) - \sin\theta\operatorname{Im}(A), \quad \theta \in \mathbb{R}.$$

Then, we also have the following upper bounds for the power of numerical radius.

Proposition 2.2. *For all $A \in B(H)$, we have the inequality*

$$\begin{aligned}w^{2p}(A) &\le 2^{3\frac{p}{2}-1}w^p((\cos\theta\operatorname{Re}(A) - \sin\theta\operatorname{Im}A)(\sin\theta\operatorname{Re}(A) + \cos\theta\operatorname{Im}(A)))\\ &\quad + 2^{p-1}\max\{\|\cos\theta\operatorname{Re}(A) - \sin\theta\operatorname{Im}(A)\|^{2p}, \|\sin\theta\operatorname{Re}(A) + \cos\theta\operatorname{Im}(A)\|^{2p}\}\\ &\le 2^{3\frac{p}{2}-1}w^p((\cos\theta\operatorname{Re}(A) - \sin\theta\operatorname{Im}A)(\sin\theta\operatorname{Re}(A) + \cos\theta\operatorname{Im}(A)))\\ &\quad + 2^{p-2}[\|\operatorname{Re}^2A + \operatorname{Im}^2A\|^p + \|\cos(2\theta)(\operatorname{Re}^2A - \operatorname{Im}^2A) - \sin(2\theta)[\operatorname{Re}(A)\operatorname{Im}(A) + \operatorname{Im}A\operatorname{Re}(A)]\|^p]\end{aligned} \tag{2.11}$$

for all $\theta \in \mathbb{R}$ and $p \ge 1$.

Proof. We use the following identity obtained in [24], see also [23, p. 34]:

$$H_{\theta+\phi} = \cos\phi H_\theta + \sin\phi H_{\theta+\frac{\pi}{2}}$$

for $\phi \in \mathbb{R}$.

From (2.7), for $\alpha = \cos\phi$, $\beta = \sin\phi$, $B = H_\theta$, and $C = H_{\theta+\frac{\pi}{2}}$, we have

$$\begin{aligned}\|H_{\theta+\phi}\|^{2p} &\le 2^{3\frac{p}{2}-1}w^p\left(H_{\theta+\frac{\pi}{2}}H_\theta\right) + 2^{p-1}\max\left\{\|H_\theta\|^{2p}, \left\|H_{\theta+\frac{\pi}{2}}\right\|^{2p}\right\}\\ &\le 2^{3\frac{p}{2}-1}w^p\left(H_{\theta+\frac{\pi}{2}}H_\theta\right) + 2^{p-2}\left[\left\|H_\theta^2 + H_{\theta+\frac{\pi}{2}}^2\right\|^p + \left\|H_\theta^2 - H_{\theta+\frac{\pi}{2}}^2\right\|^p\right]\end{aligned}$$

for $\theta, \phi \in \mathbb{R}$.

If we take the supremum over $\phi \in \mathbb{R}$, we obtain

$$\begin{aligned}w^{2p}(A) = \sup_{\phi\in\mathbb{R}}\|H_{\theta+\phi}\|^{2p} &\le 2^{3\frac{p}{2}-1}w^p\left(H_{\theta+\frac{\pi}{2}}H_\theta\right) + 2^{p-1}\max\left\{\|H_\theta\|^{2p}, \left\|H_{\theta+\frac{\pi}{2}}\right\|^{2p}\right\}\\ &\le 2^{3\frac{p}{2}-1}w^p\left(H_{\theta+\frac{\pi}{2}}H_\theta\right) + 2^{p-2}\left[\left\|H_\theta^2 + H_{\theta+\frac{\pi}{2}}^2\right\|^p + \left\|H_\theta^2 - H_{\theta+\frac{\pi}{2}}^2\right\|^p\right].\end{aligned} \tag{2.12}$$

Observe that

$$\begin{aligned}
H_{\theta+\frac{\pi}{2}} &= \cos\left(\theta+\frac{\pi}{2}\right)\operatorname{Re}A - \sin\left(\theta+\frac{\pi}{2}\right)\operatorname{Im}(A) = -\sin\theta\operatorname{Re}A - \cos\theta\operatorname{Im}(A),\\
H_\theta^2 &= (\cos\theta\operatorname{Re}(A) - \sin\theta\operatorname{Im}(A))^2\\
&= \cos^2\theta\operatorname{Re}^2A - \sin\theta\cos\theta[\operatorname{Re}(A)\operatorname{Im}(A) + \operatorname{Im}(A)\operatorname{Re}(A)] + \sin^2\theta\operatorname{Im}^2A,\\
H_{\theta+\frac{\pi}{2}}^2 &= (\sin\theta\operatorname{Re}(A) + \cos\theta\operatorname{Im}(A))^2\\
&= \sin^2\theta\operatorname{Re}^2A + \sin\theta\cos\theta[\operatorname{Re}(A)\operatorname{Im}(A) + \operatorname{Im}(A)\operatorname{Re}(A)] + \cos^2\theta\operatorname{Im}^2A,
\end{aligned}$$

which gives that

$$H_\theta^2 + H_{\theta+\frac{\pi}{2}}^2 = \operatorname{Re}^2A + \operatorname{Im}^2A$$

and

$$\begin{aligned}
H_\theta^2 - H_{\theta+\frac{\pi}{2}}^2 &= (\cos^2\theta - \sin^2\theta)\operatorname{Re}^2A + (\sin^2\theta - \cos^2\theta)\operatorname{Im}^2A - 2\sin\theta\cos\theta[\operatorname{Re}(A)\operatorname{Im}(A) + \operatorname{Im}(A)\operatorname{Re}(A)]\\
&= \cos(2\theta)(\operatorname{Re}^2A - \operatorname{Im}^2A) - \sin(2\theta)[\operatorname{Re}(A)\operatorname{Im}(A) + \operatorname{Im}(A)\operatorname{Re}(A)]
\end{aligned}$$

for $\theta \in \mathbb{R}$ and by (2.12) we derive (2.11). □

The case when $\theta = \frac{\pi}{4}$ is of interest and is incorporated in:

Corollary 2.3. *For all $A \in B(H)$, we have the numerical radius inequality*

$$\begin{aligned}
w^{2p}(A) &\le 2^{\frac{p}{2}-1}w^p((\operatorname{Re}(A) - \operatorname{Im}(A))(\operatorname{Re}(A) + \operatorname{Im}(A)))\\
&\quad + \frac{1}{2}\max\{\|\operatorname{Re}(A) - \operatorname{Im}(A)\|^{2p}, \|\operatorname{Re}(A) + \operatorname{Im}(A)\|^{2p}\}\\
&\le 2^{\frac{p}{2}-1}w^p((\operatorname{Re}(A) - \operatorname{Im}(A))(\operatorname{Re}(A) + \operatorname{Im}(A)))\\
&\quad + 2^{p-2}[\|\operatorname{Re}^2A + \operatorname{Im}^2A\|^p + \|\operatorname{Re}(A)\operatorname{Im}(A) + \operatorname{Im}(A)\operatorname{Re}(A)\|^p]
\end{aligned} \tag{2.13}$$

for $p \ge 1$.

For $p = 1$ in (2.13), we obtain

$$\begin{aligned}
w^2(A) &\le \frac{\sqrt{2}}{2}w((\operatorname{Re}(A) - \operatorname{Im}(A))(\operatorname{Re}(A) + \operatorname{Im}(A)))\\
&\quad + \frac{1}{2}\max\{\|\operatorname{Re}(A) - \operatorname{Im}(A)\|^2, \|\operatorname{Re}(A) + \operatorname{Im}(A)\|^2\}\\
&\le \frac{\sqrt{2}}{2}w((\operatorname{Re}(A) - \operatorname{Im}(A))(\operatorname{Re}(A) + \operatorname{Im}(A)))\\
&\quad + \frac{1}{2}[\|\operatorname{Re}^2A + \operatorname{Im}^2A\| + \|\operatorname{Re}(A)\operatorname{Im}(A) + \operatorname{Im}(A)\operatorname{Re}(A)\|].
\end{aligned}$$

Finally, for this section, we also have

Proposition 2.3. *For all $A \in B(H)$, the following numerical radius inequality holds*

$$w^{2p}(A) \le \frac{1}{2}\left(\frac{1}{2}\sqrt{2^{\frac{p}{2}}w^p(A^4) + \|A^2\|^{2p}} + \left\|\frac{|A|^2 + |A^*|^2}{2}\right\|^p\right), \tag{2.14}$$

where $p \ge 1$.

Proof. Since

$$H_\theta \coloneqq \operatorname{Re}(e^{i\theta}A) = \frac{1}{2}(e^{i\theta}A + e^{-i\theta}A^*),$$

we see that

$$H_\theta^2 = \frac{1}{4}(e^{2i\theta}A^2 + e^{-2i\theta}(A^*)^2 + |A|^2 + |A^*|^2)$$

for $\theta \in \mathbb{R}$.

Then, by taking the norm and the power $p \geq 1$, we obtain, by the convexity of the power function, that

$$\begin{aligned} \|H_\theta^2\|^p &= \left\| \frac{1}{2}\left(\frac{e^{2i\theta}A^2 + e^{-2i\theta}(A^*)^2}{2} + \frac{|A|^2 + |A^*|^2}{2} \right) \right\|^p \\ &\leq \frac{1}{2}\left(\left\| \frac{e^{2i\theta}A^2 + e^{-2i\theta}(A^*)^2}{2} \right\|^p + \left\| \frac{|A|^2 + |A^*|^2}{2} \right\|^p \right). \end{aligned} \tag{2.15}$$

From (2.7), by taking the square root, we obtain for $p \geq 1$ that

$$\|\alpha B + \beta C\|^p \leq (|\alpha|^2 + |\beta|^2)^{\frac{p}{2}} 2^{\frac{p-1}{2}} \sqrt{2^{\frac{p}{2}} w^p(C^*B) + \max\{\|B\|^{2p}, \|C\|^{2p}\}} \tag{2.16}$$

for $B, C \in B(H)$ and $\alpha, \beta \in \mathbb{C}$.

Using (2.16) for $\alpha = \frac{e^{2i\theta}}{2}$, $\beta = \frac{e^{-2i\theta}}{2}$, $B = A^2$, and $C = (A^*)^2$, we obtain

$$\left\| \frac{e^{2i\theta}A^2 + e^{-2i\theta}(A^*)^2}{2} \right\|^p \leq \left(\frac{1}{2}\right)^{\frac{p}{2}} 2^{\frac{p-1}{2}} \sqrt{2^{\frac{p}{2}} w^p(A^4) + \|A^2\|^{2p}} = \frac{1}{2}\sqrt{2^{\frac{p}{2}} w^p(A^4) + \|A^2\|^{2p}}$$

and by (2.15) we derive

$$\|H_\theta^2\|^p \leq \frac{1}{2}\left(\frac{1}{2}\sqrt{2^{\frac{p}{2}} w^p(A^4) + \|A^2\|^{2p}} + \left\| \frac{|A|^2 + |A^*|^2}{2} \right\|^p \right).$$

By taking the supremum over $\theta \in \mathbb{R}$, we deduce the desired result (2.14). □

For $p = 1$, we obtain

$$w^2(A) \leq \frac{1}{2}\left(\frac{1}{2}\sqrt{\sqrt{2}\, w(A^4) + \|A^2\|^2} + \left\| \frac{|A|^2 + |A^*|^2}{2} \right\| \right).$$

3 Applications of norm and numerical radius inequalities

In this section, we present several power inequalities for the s-r-norm and s-r-numerical radius. By considering $(B, C) = (A, A^*)$ or $(B, C) = (\operatorname{Re}(A), \operatorname{Im}(A))$ for $A \in B(H)$, we derive norm and numerical radius inequalities for a single operator.

Our first result reads as follows.

Theorem 3.1. *Let $B, C \in B(H)$, then for $p \geq 1$ and $r > 1$*

$$\begin{aligned} \|(B, C)\|_r^{2pr} &\leq 2^{p-1} \|(B, C)\|_{2(r-1)}^{2p(r-1)} \left[2^{\frac{p}{2}} w^p(C^*B) + \max\{\|B\|^{2p}, \|C\|^{2p}\} \right] \\ &\leq 2^{p-1} \|(B, C)\|_{2(r-1)}^{2p(r-1)} \left[2^{\frac{p}{2}} w^p(C^*B) + \frac{1}{2}(\||B|^2 + |C|^2\|^p + \||B|^2 - |C|^2\|^p) \right] \end{aligned} \tag{3.1}$$

while for $r = 1$,

$$\begin{aligned}\|(B,C)\|^{2p} &\le 2^{2p-1}\left[2^{\frac{p}{2}}w^p(C^*B) + \max\{\|B\|^{2p}, \|C\|^{2p}\}\right]\\ &\le 2^{2p-1}\left[2^{\frac{p}{2}}w^p(C^*B) + \frac{1}{2}(\||B|^2 + |C|^2\|^p + \||B|^2 - |C|^2\|^p)\right].\end{aligned} \tag{3.2}$$

Proof. If we take, for $r > 1$,

$$\alpha := \begin{cases}\langle x, By\rangle|\langle x, By\rangle|^{r-2} & \text{if } \langle x, By\rangle \neq 0\\ 0 & \text{if } \langle x, By\rangle \neq 0\end{cases}$$

and

$$\beta := \begin{cases}\langle x, Cy\rangle|\langle x, Cy\rangle|^{r-2} & \text{if } \langle x, Cy\rangle \neq 0\\ 0 & \text{if } \langle x, Cy\rangle \neq 0,\end{cases}$$

then for $r \ge 1$,

$$|\alpha| = |\langle x, By\rangle|^{r-1} \quad \text{and} \quad |\beta| = |\langle x, Cy\rangle|^{r-1}.$$

Also,

$$|\alpha|^2 = |\langle x, By\rangle|^{2(r-1)}, \quad |\beta|^2 = |\langle x, Cy\rangle|^{2(r-1)}$$

and

$$|\langle x, (\alpha B + \beta C)y\rangle| = |\overline{\alpha}\langle x, By\rangle + \overline{\beta}\langle x, Cy\rangle| = |\langle x, By\rangle|^r + |\langle x, Cy\rangle|^r$$

for $r \ge 1$.

From (2.2), we obtain the vector inequality

$$\begin{aligned}&[|\langle x, By\rangle|^r + |\langle x, Cy\rangle|^r]^{2p}\\ &\le 2^{3\frac{p}{2}-1}\|x\|^{2p}(|\langle x, By\rangle|^{2(r-1)} + |\langle x, Cy\rangle|^{2(r-1)})^p|\langle C^*By, y\rangle|^p\\ &\quad + 2^{p-1}\|x\|^{2p}(|\langle x, By\rangle|^{2(r-1)} + |\langle x, Cy\rangle|^{2(r-1)})^p\max\{\|By\|^{2p}, \|Cy\|^{2p}\},\\ &\le 2^{3\frac{p}{2}-1}\|x\|^{2p}(|\langle x, By\rangle|^{2(r-1)} + |\langle x, Cy\rangle|^{2(r-1)})^p|\langle C^*By, y\rangle|^p\\ &\quad + 2^{p-2}\|x\|^{2p}(|\langle x, By\rangle|^{2(r-1)} + |\langle x, Cy\rangle|^{2(r-1)})^p\\ &\quad \times [\langle(|B|^2 + |C|^2)y, y\rangle^p + |\langle(|B|^2 - |C|^2)y, y\rangle|^p]\end{aligned} \tag{3.3}$$

for all $x, y \in H$ and $r \ge 1$.

By taking the supremum in (3.3) over $\|x\| = \|y\| = 1$ and observing that, for $r > 1$,

$$\sup_{\|x\|=\|y\|=1}[|\langle x, By\rangle|^r + |\langle x, Cy\rangle|^r]^{2p} = \|(B,C)\|_r^{2pr},$$

$$\begin{aligned}&\sup_{\|x\|=\|y\|=1}[\|x\|^{2p}(|\langle x, By\rangle|^{2(r-1)} + |\langle x, Cy\rangle|^{2(r-1)})^p|\langle C^*By, y\rangle|^p]\\ &\le \sup_{\|x\|=\|y\|=1}(|\langle x, By\rangle|^{2(r-1)} + |\langle x, Cy\rangle|^{2(r-1)})^p\sup_{\|y\|=1}|\langle C^*By, y\rangle|^p\\ &= \|(B,C)\|_{2(r-1)}^{2p(r-1)}w^p(C^*B)\end{aligned}$$

and

$$\begin{aligned}
&\sup_{\|x\|=\|y\|=1} [\|x\|^{2p}(|\langle x, By\rangle|^{2(r-1)} + |\langle x, Cy\rangle|^{2(r-1)})^{p} \\
&\qquad \times \left\{\begin{array}{l} \max\{\|By\|^{2p}, \|Cy\|^{2p}\}], \\ \left.\dfrac{\langle(|B|^2 + |C|^2)y, y\rangle^{p} + |\langle(|B|^2 - |C|^2)y, y\rangle|^{p}}{2}\right] \end{array}\right. \\
&\leq \sup_{\|x\|=\|y\|=1} (|\langle x, By\rangle|^{2(r-1)} + |\langle x, Cy\rangle|^{2(r-1)})^{p} \\
&\qquad \times \left\{\begin{array}{l} \sup\limits_{\|y\|=1} \max\{\|By\|^{2p}, \|Cy\|^{2p}\} \\ \\ \sup\limits_{\|y\|=1} \left[\dfrac{\langle(|B|^2 + |C|^2)y, y\rangle^{p} + |\langle(|B|^2 - |C|^2)y, y\rangle|^{p}}{2}\right] \end{array}\right. \\
&\leq \|(B, C)\|_{2(r-1)}^{2p(r-1)} \left\{\begin{array}{l} \max\{\|B\|^{2p}, \|C\|^{2p}\}, \\ \dfrac{\||B|^2 + |C|^2\|^{p} + \||B|^2 - |C|^2\|^{p}}{2} \end{array}\right.,
\end{aligned}$$

we obtain the desired result (3.1).

From (3.3), for $r = 1$, we have the vector inequality

$$\begin{aligned}
[|\langle x, By\rangle| + |\langle x, Cy\rangle|]^{2p} &\leq 2^{2p-1}\, \|x\|^{2p}(|\langle C^{*}By, y\rangle|^{p} + \max\{\|By\|^{2p}, \|Cy\|^{2p}\}) \\
&\leq 2^{2p-1}\, \|x\|^{2p}\left(|\langle C^{*}By, y\rangle|^{p} + \frac{\langle(|B|^2 + |C|^2)y, y\rangle^{p} + |\langle(|B|^2 - |C|^2)y, y\rangle|^{p}}{2}\right)
\end{aligned}$$

for $x, y \in H$, which by the same argument gives inequality (3.2). ☐

By taking $r = 2$ in (3.1), we derive the following corollary.

Corollary 3.1. *Let $B, C \in B(H)$, then for $p \geq 1$*

$$\begin{aligned}
\|(B, C)\|_{e}^{2p} &\leq 2^{p-1}\left[2^{\frac{p}{2}} w^{p}(C^{*}B) + \max\{\|B\|^{2p}, \|C\|^{2p}\}\right] \\
&\leq 2^{p-1}\left[2^{\frac{p}{2}} w^{p}(C^{*}B) + \frac{\||B|^2 + |C|^2\|^{p} + \||B|^2 - |C|^2\|^{p}}{2}\right].
\end{aligned}$$

Several consequences of Theorem 3.1 may be derived in the next remark.

Remark 3.1. (1) If we take $p = 1$ in Theorem 3.1, then we obtain for $r > 1$, that

$$\begin{aligned}
\|(B, C)\|_{r}^{2r} &\leq \|(B, C)\|_{2(r-1)}^{2(r-1)}[\sqrt{2}\, w(C^{*}B) + \max\{\|B\|^{2}, \|C\|^{2}\}] \\
&\leq \|(B, C)\|_{2(r-1)}^{2(r-1)}\left[\sqrt{2}\, w(C^{*}B) + \frac{\||B|^2 + |C|^2\| + \||B|^2 - |C|^2\|}{2},\right],
\end{aligned}$$

while for $r = 1$,

$$\begin{aligned}
\|(B, C)\|^{2} &\leq 2[\sqrt{2}\, w(C^{*}B) + \max\{\|B\|^{2}, \|C\|^{2}\}] \\
&\leq 2\left[\sqrt{2}\, w(C^{*}B) + \frac{\||B|^2 + |C|^2\| + \||B|^2 - |C|^2\|}{2}\right].
\end{aligned}$$

(2) Let $A \in B(H)$ and if we take $B = A$ and $C = A^{*}$, and put

$$\delta_{r}(A) \coloneqq \|(A, A^{*})\|_{r} = \sup_{\|x\|=\|y\|=1} (|\langle x, Ay\rangle|^{r} + |\langle x, A^{*}y\rangle|^{r})^{\frac{1}{r}} \leq 2^{\frac{1}{r}}\|A\| \quad \text{for } r \geq 1,$$

then by Theorem 3.1, we have for $r > 1$ that

$$\begin{aligned}\delta_r^{2pr}(A) &\leq 2^{p-1}\delta_{2(r-1)}^{2p(r-1)}(A)\left[2^{\frac{p}{2}}w^p(A^2) + \|A\|^{2p}\right]\\ &\leq 2^{p-1}\delta_{2(r-1)}^{2p(r-1)}(A)\left[2^{\frac{p}{2}}w^p(A^2) + \frac{\||A|^2 + |A^*|^2\|^p + \||A|^2 - |A^*|^2\|^p}{2}\right]\end{aligned} \tag{3.4}$$

while for $r = 1$,

$$\begin{aligned}\delta_1^{2p}(A) &\leq 2^{2p-1}\left[2^{\frac{p}{2}}w^p(A^2) + \|A\|^{2p}\right]\\ &\leq 2^{2p-1}\left[2^{\frac{p}{2}}w^p(A^2) + \frac{\||A|^2 + |A^*|^2\|^p + \||A|^2 - |A^*|^2\|^p}{2}\right].\end{aligned}$$

For $r = 2$, we have

$$\delta_e(A) \coloneqq \|(A, A^*)\|_2 = \sup_{\|x\|=\|y\|=1} (|\langle x, Ay\rangle|^2 + |\langle x, A^*y\rangle|^2)^{\frac{1}{2}} \leq \sqrt{2}\,\|A\|.$$

If we take $r = 2$ in (3.4), then we obtain

$$\begin{aligned}\delta_e^{2p}(A) &\leq 2^{p-1}\left[2^{\frac{p}{2}}w^p(A^2) + \|A\|^{2p}\right]\\ &\leq 2^{p-1}\left[2^{\frac{p}{2}}w^p(A^2) + \frac{\||A|^2 + |A^*|^2\|^p + \||A|^2 - |A^*|^2\|^p}{2}\right].\end{aligned}$$

We also have the numerical radius inequalities.

Theorem 3.2. *Let $B, C \in B(H)$, then for $p \geq 1$ and $r > 1$*

$$\begin{aligned}w_r^{2pr}(B, C) &\leq 2^{p-1}w_{2(r-1)}^{2p(r-1)}(B, C)\left[2^{\frac{p}{2}}w^p(C^*B) + \max\{\|B\|^{2p}, \|C\|^{2p}\}\right]\\ &\leq 2^{p-1}w_{2(r-1)}^{2p(r-1)}(B, C)\left[2^{\frac{p}{2}}w^p(C^*B) + \frac{\||B|^2 + |C|^2\|^p + \||B|^2 - |C|^2\|^p}{2}\right]\end{aligned} \tag{3.5}$$

while for $r = 1$,

$$\begin{aligned}w^{2p}(B, C) &\leq 2^{2p-1}\left[2^{\frac{p}{2}}w^p(C^*B) + \max\{\|B\|^{2p}, \|C\|^{2p}\}\right]\\ &\leq 2^{2p-1}\left[2^{\frac{p}{2}}w^p(C^*B) + \frac{\||B|^2 + |C|^2\|^p + \||B|^2 - |C|^2\|^p}{2}\right].\end{aligned} \tag{3.6}$$

Proof. From (3.3), for $y = x$, we have for $r > 1$ that

$$\begin{aligned}[|\langle x, Bx\rangle|^r + |\langle x, Cx\rangle|^r]^{2p} \leq\ & 2^{3\frac{p}{2}-1}\|x\|^{2p}(|\langle x, Bx\rangle|^{2(r-1)} + |\langle x, Cx\rangle|^{2(r-1)})^p|\langle C^*Bx, x\rangle|^p\\ &+ 2^{p-1}\|x\|^{2p}(|\langle x, Bx\rangle|^{2(r-1)} + |\langle x, Cx\rangle|^{2(r-1)})^p \max\{\|Bx\|^{2p}, \|Cx\|^{2p}\}\\ \leq\ & 2^{3\frac{p}{2}-1}\|x\|^{2p}(|\langle x, Bx\rangle|^{2(r-1)} + |\langle x, Cx\rangle|^{2(r-1)})^p|\langle C^*Bx, x\rangle|^p\\ &+ 2^{p-1}\|x\|^{2p}(|\langle x, Bx\rangle|^{2(r-1)} + |\langle x, Cx\rangle|^{2(r-1)})^p\\ &\times \frac{\langle(|B|^2 + |C|^2)x, x\rangle^p + |\langle(|B|^2 - |C|^2)x, x\rangle|^p}{2}\end{aligned}$$

and for $r = 1$, that

$$\begin{aligned}[|\langle x, Bx\rangle| + |\langle x, Cx\rangle|]^{2p} \leq\ & 2^{2p-1}\|x\|^{2p}(|\langle C^*Bx, x\rangle|^p + \max\{\|Bx\|^{2p}, \|Cx\|^{2p}\})\\ \leq\ & 2^{2p-1}\|x\|^{2p} \times \left(|\langle C^*Bx, x\rangle|^p + \frac{\langle(|B|^2 + |C|^2)x, x\rangle^p + |\langle(|B|^2 - |C|^2)x, x\rangle|^p}{2}\right)\end{aligned}$$

for all $x \in H$.

By taking the supremum over $\|x\| = 1$, we obtain, as above, the desired inequalities (3.5) and (3.6). □

Corollary 3.2. *Let $B, C \in B(H)$, then for $p \geq 1$*

$$\begin{aligned} w_e^{2p}(B,C) &\leq 2^{p-1}\left[2^{\frac{p}{2}} w^p(C^*B) + \max\{\|B\|^{2p}, \|C\|^{2p}\}\right] \\ &\leq 2^{p-1}\left[2^{\frac{p}{2}} w^p(C^*B) + \frac{\||B|^2 + |C|^2\|^p + \||B|^2 - |C|^2\|^p}{2}\right]. \end{aligned}$$

Remark 3.2. (1) If we take $p = 1$ in Theorem 3.2, then we obtain for $r > 1$ that

$$\begin{aligned} w_r^{2r}(B,C) &\leq w_{2(r-1)}^{2(r-1)}(B,C)[\sqrt{2}\, w(C^*B) + \max\{\|B\|^2, \|C\|^2\}] \\ &\leq w_{2(r-1)}^{2(r-1)}(B,C)\left[\sqrt{2}\, w(C^*B) + \frac{\||B|^2 + |C|^2\| + \||B|^2 - |C|^2\|}{2}\right], \end{aligned}$$

while for $r = 1$,

$$\begin{aligned} w^2(B,C) &\leq 2[\sqrt{2}\, w(C^*B) + \max\{\|B\|^2, \|C\|^2\}] \\ &\leq 2\left[\sqrt{2}\, w(C^*B) + \frac{\||B|^2 + |C|^2\| + \||B|^2 - |C|^2\|}{2}\right]. \end{aligned}$$

(2) For $r \geq 1$ and $B = A$, $C = A^*$, we have that

$$w_r(A, A^*) := \sup_{\|x\|=1} (|\langle x, Ax\rangle|^r + |\langle x, A^*x\rangle|^r)^{\frac{1}{r}} = 2^{\frac{1}{r}} w(A).$$

From (3.5), we then obtain

$$\begin{aligned} w^{2p}(A) &\leq 2^{p-1}\left[2^{\frac{p}{2}} w^p(A^2) + \|A\|^{2p}\right] \\ &\leq 2^{p-1}\left[2^{\frac{p}{2}} w^p(A^2) + \frac{\||A|^2 + |A^*|^2\|^p + \||A|^2 - |A^*|^2\|^p}{2}\right] \end{aligned}$$

for $p \geq 1$.

(3) Let $A \in B(H)$ and $A = \mathrm{Re}(A) + i\,\mathrm{Im}(A)$ be its Cartesian decomposition. For $r \geq 1$, we can introduce the quantities

$$\eta_r(A) := \|(\mathrm{Re}(A), \mathrm{Im}(A))\|_r = \sup_{\|x\|=\|y\|=1} (|\langle x, (\mathrm{Re}(A))y\rangle|^r + |\langle x, (\mathrm{Im}(A))y\rangle|^r)^{\frac{1}{r}}$$

and

$$\rho_r(A) := w_r(\mathrm{Re}(A), \mathrm{Im}(A)) = \sup_{\|x\|=1} (|\langle x, (\mathrm{Re}(A))x\rangle|^r + |\langle x, (\mathrm{Im}(A))x\rangle|^r)^{\frac{1}{r}}.$$

For $r = 1$, we simply write $\eta(A)$ and $\rho(A)$.

For $r = 2$, we note that

$$\begin{aligned} \eta_e(A) := \|(\mathrm{Re}(A), \mathrm{Im}(A))\|_e &= \sup_{\|x\|=\|y\|=1} (|\langle x, (\mathrm{Re}(A))y\rangle|^2 + |\langle x, (\mathrm{Im}(A))y\rangle|^2)^{\frac{1}{2}} \\ &= \sup_{\|x\|=\|y\|=1} |\langle x, Ay\rangle| = \|A\| \end{aligned}$$

and

$$\begin{aligned} \rho_e(A) := w_e(\mathrm{Re}(A), \mathrm{Im}(A)) &= \sup_{\|x\|=1} (|\langle x, (\mathrm{Re}(A))x\rangle|^2 + |\langle x, (\mathrm{Im}(A))x\rangle|^2)^{\frac{1}{2}} \\ &= \sup_{\|x\|=1} |\langle x, Ax\rangle| = w(A). \end{aligned}$$

Remark 3.3. (1) If we substitute in Theorem 3.1 $B = \operatorname{Re}(A)$ and $C = \operatorname{Im}(A)$, then for $p \geq 1$ and $r > 1$, we obtain

$$\begin{aligned}\eta_r^{2pr}(A) &\leq 2^{p-1}\eta_{2(r-1)}^{2p(r-1)}(A)\left[2^{\frac{p}{2}}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + \max\{\|\operatorname{Re}(A)\|^{2p}, \|\operatorname{Im}(A)\|^{2p}\}\right]\\ &\leq 2^{p-1}\eta_{2(r-1)}^{2p(r-1)}(A)\left[2^{\frac{p}{2}}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + \frac{\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\|^p + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|^p}{2}\right],\end{aligned}$$

while for $r = 1$,

$$\begin{aligned}\eta^{2pr}(A) &\leq 2^{2p-1}\left[2^{\frac{p}{2}}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + \max\{\|\operatorname{Re}(A)\|^{2p}, \|\operatorname{Im}(A)\|^{2p}\}\right]\\ &\leq 2^{2p-1}\left[2^{\frac{p}{2}}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + \frac{\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\|^p + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|^p}{2}\right].\end{aligned}$$

For $r = 2$, we obtain

$$\begin{aligned}\|A\|^{2p} &\leq 2^{p-1}\left[2^{\frac{p}{2}}\omega^p(\operatorname{Im}(A)\operatorname{Re}(A)) + \max\{\|\operatorname{Re}(A)\|^{2p}, \|\operatorname{Im}(A)\|^{2p}\}\right]\\ &\leq 2^{p-1}\left[2^{\frac{p}{2}}\omega^p(\operatorname{Im}(A)\operatorname{Re}(A)) + \frac{\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\|^p + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|^p}{2}\right].\end{aligned}$$

(2) If we use Theorem 3.2 for $B = \operatorname{Re}(A)$ and $C = \operatorname{Im}(A)$, then for $p \geq 1$ and $r > 1$, we obtain

$$\begin{aligned}\rho_r^{2pr}(A) &\leq 2^{p-1}\rho_{2(r-1)}^{2p(r-1)}(A)\left[2^{\frac{p}{2}}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + \max\{\|\operatorname{Re}(A)\|^{2p}, \|\operatorname{Im}(A)\|^{2p}\}\right]\\ &\leq 2^{p-1}\rho_{2(r-1)}^{2p(r-1)}(A)\left[2^{\frac{p}{2}}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + \frac{\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\|^p + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|^p}{2}\right],\end{aligned}$$

while for $r = 1$,

$$\begin{aligned}\rho^{2p}(A) &\leq 2^{2p-1}\left[2^{\frac{p}{2}}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + \max\{\|\operatorname{Re}(A)\|^{2p}, \|\operatorname{Im}(A)\|^{2p}\}\right]\\ &\leq 2^{2p-1}\left[2^{\frac{p}{2}}w^p(\operatorname{Im}(A)\operatorname{Re}(A)) + \frac{\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\|^p + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|^p}{2}\right].\end{aligned}$$

For $r = 2$, we recapture (2.9). The case $p = 1$ gives for $r > 1$ that

$$\begin{aligned}\rho_r^{2r}(A) &\leq \rho_{2(r-1)}^{2(r-1)}(A)[\sqrt{2}\,w(\operatorname{Im}(A)\operatorname{Re}(A)) + \max\{\|\operatorname{Re}(A)\|^2, \|\operatorname{Im}(A)\|^2\}]\\ &\leq \rho_{2(r-1)}^{2(r-1)}(A)\left[\sqrt{2}\,w(\operatorname{Im}(A)\operatorname{Re}(A)) + \frac{\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\| + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|}{2}\right],\end{aligned}$$

while for $r = 1$,

$$\begin{aligned}\rho^2(A) &\leq 2[\sqrt{2}\,w(\operatorname{Im}(A)\operatorname{Re}(A)) + \max\{\|\operatorname{Re}(A)\|^2, \|\operatorname{Im}(A)\|^2\}]\\ &\leq 2\left[\sqrt{2}\,w(\operatorname{Im}(A)\operatorname{Re}(A)) + \frac{\|\operatorname{Re}^2(A) + \operatorname{Im}^2(A)\| + \|\operatorname{Re}^2(A) - \operatorname{Im}^2(A)\|}{2}\right],\end{aligned}$$

For $r = 2$, we re-obtain (2.10).

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