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Explicit expressions for one family of the four parametric linear Euler sums

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ABSTRACT

In this paper, we investigate one in the family of four parametric linear Euler sums as given by Alzer and Choi and later extended by Sofo and Choi. We give explicit expressions for a variety of special cases involving a free parameter. Our analysis incorporates the properties of the polylogarithm function, commonly referred to as the Fermi–Dirac integral. A reciprocity property is utilized to highlight an explicit representation of a particular skew harmonic linear Euler sum. A number of examples are also given which highlight the theorems. This work generalizes some results in the published literature.

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1. Introduction and notation

Euler and Goldbach have investigated linear harmonic Euler sums of the type

$$\mathbb{S}_{p,t} = \sum_{n \geq 1} \frac{H_n^{(p)}}{n^t} \quad (1.1)$$

with weight $p + t$, see Flajolet and Salvy [1]. Many researchers including Nielsen [2] and others, see Refs [1,3–7], continued this work and it is now known that $\mathbb{S}_{p,t}$ can be explicitly evaluated, in terms of special functions such as the Riemann zeta function, in the cases when $p = t \in \mathbb{N}$, $p + t$ of odd weight, $p + t$ of even weight in only the pair $\{(4, 2), (2, 4)\}$ with $p \neq t$. A shuffle (reciprocity) relation

$$\mathbb{S}_{p,t} + \mathbb{S}_{t,p} = \zeta(p) \zeta(t) + \zeta(p + t)$$

exists to evaluate $\mathbb{S}_{t,p}$ in the case $\mathbb{S}_{p,t}$ is known (or visa-versa). The subsequent notion of four distinct classes of linear harmonic Euler sums, of the kind

$$\mathbb{S}_{p,t}^{++}(a, b; q) := \sum_{n=1}^{\infty} \frac{H_{qn}^{(p)}(a)}{(n+b)^t}, \quad \mathbb{S}_{p,t}^{+-}(a, b; q) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_{qn}^{(p)}(a)}{(n+b)^t},$$

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$$\mathbb{S}_{p,t}^{-+}(a,b;q) := \sum_{n=1}^{\infty} \frac{A_{qn}^{(p)}(a)}{(n+b)^t}, \quad \mathbb{S}_{p,t}^{--}(a,b;q) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{A_{qn}^{(p)}(a)}{(n+b)^t}, \quad (1.2)$$

where $q \in \mathbb{Z}_{\geq 1}$, $a, b \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$ and $p, t \in \mathbb{C} \setminus \mathbb{Z}^-$ was identified by Flajolet and Salvy [1], in the case $a = 0, b = 0$ and $q = 1$. The case $a \in \mathbb{Z}_{\geq 1}, b \in \mathbb{Z}_{\geq 1}$ and $q = 1$ was investigated by Alzer and Choi [3], further investigated by Rui et al. [8] and finally the case $a \in \mathbb{Z}_{\geq 1}, b \in \mathbb{Z}_{\geq 1}$ and $q \in \mathbb{Z}_{\geq 1}$ was examined by Sofo and Choi [9]. The skew linear harmonic Euler sums (1.4) for certain values of the parameters a, p and t belong to one of the family (1.2), namely, the family

$$\mathbb{S}_{t,p+2}^{-+}\left(0, \frac{1+a}{2}; 1\right) := \mathbb{S}_{t,p+2}^{-+}\left(0, \frac{1+a}{2}\right) = \sum_{n=1}^{\infty} \frac{A_n^{(t)}}{\left(n + \frac{1+a}{2}\right)^{p+2}} \quad (1.3)$$

and its composition,

$$\sum_{n \geq 1} A_n^{(t)} \left\{ \frac{1}{(2n+1+a)^{p+1}} + \frac{(-1)^{p+t+1}}{(2n+1-a)^{p+1}} \right\} \quad (1.4)$$

will be explicitly represented, in this paper, in terms of special functions. Using a reciprocity theorem due to Alzer and Choi [3], we also represent

$$\mathbb{S}_{p+2,t}^{+-}\left(\frac{1+a}{2}, 0\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_n^{(p+2)}\left(\frac{1+a}{2}\right)}{n^t} \quad (1.5)$$

explicitly, in terms of special functions. We also obtain the new identity, see (4.6)

$$\begin{aligned} \mathbb{S}_{p+2,t}^{+-}(0, 0, 2) &= \sum_{n \geq 1} \frac{(-1)^{n+1} H_{2n}^{(p+2)}}{n^t} = \eta(t) \left(\eta(p+2) + \frac{\zeta(p+2)}{2^{p+2}} \right) \\ &+ \frac{1}{2^{p+2}} \left(\mathbb{S}_{p+2,t}^{+-} - \mathbb{S}_{t,p+2}^{-+}\left(0, \frac{1}{2}\right) \right). \end{aligned}$$

Interest in Euler sums and multiple zeta values has recently been topics of popular study by many authors (see, for example, [10–16]). Likewise, various Euler sums with parameters have been researched (see, for example, Refs [17–21]). We now define some notation which will be useful throughout the remainder of the paper. The generalized harmonic numbers $H_n^{(s)}(u)$ of order s are defined by

$$H_n^{(s)}(u) := \sum_{j=1}^n \frac{1}{(j+u)^s} \quad (s \in \mathbb{C}, u \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}, n \in \mathbb{Z}_{\geq 1}), \quad (1.6)$$

and $H_n^{(s)} := H_n^{(s)}(0)$ are the harmonic numbers of order s . The harmonic numbers $H_n := H_n^{(1)}$ are given by

$$H_n = \sum_{j=1}^n \frac{1}{j} = \gamma + \psi(n+1) \quad (n \in \mathbb{Z}_{\geq 0}) \quad \text{and} \quad H_0 := 0. \quad (1.7)$$

The term γ represents the familiar Euler-Mascheroni constant (see, e.g. [20, Section 1.2]) and $\psi(z)$ denotes the digamma (or psi) function defined by

$$\psi(z) := \frac{d}{dz} (\log \Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}), \quad (1.8)$$

where $\Gamma(z)$ is the familiar Gamma function (see, e.g. [20, Section 1.1]). There are many identities involving the psi function, the following one is recalled (see, e.g. [20, Section 1.3]):

$$\psi(z+n) = \psi(z) + \sum_{k=1}^n \frac{1}{z+k-1} \quad (n \in \mathbb{Z}_{\geq 1}). \quad (1.9)$$

In this paper, an empty sum is assumed to be nil. We also, let \mathbb{C} , \mathbb{R} , and \mathbb{Z} stand for the sets of complex numbers, real numbers, and integers, respectively. Also let $S_{>\ell}$, $S_{\geq\ell}$, $S_{<\ell}$, and $S_{\leq\ell}$ be the subsets of the set S (\mathbb{R} or \mathbb{Z}) which are greater than, greater than or equal to, less than, and less than or equal to some $\ell \in \mathbb{R}$, respectively. Particularly, put $\mathbb{N} := \mathbb{Z}_{\geq 1}$. The polygamma function $\psi^{(k)}(z)$ defined by

$$\begin{aligned} \psi^{(k)}(z) &:= \frac{d^k}{dz^k} \{\psi(z)\} = (-1)^{k+1} k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}} = (-1)^{k+1} k! \zeta(k+1, z), \\ &(k \in \mathbb{N}; z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}). \end{aligned} \quad (1.10)$$

The generalized (or Hurwitz) zeta function, $\zeta(s, z)$ is defined by

$$\zeta(s, z) = \sum_{m=0}^{\infty} \frac{1}{(m+z)^s} \quad (\Re(s) > 1, z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}). \quad (1.11)$$

The Dirichlet eta function, $\eta(s)$ is defined by

$$\eta(s) := \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s}) \zeta(s) \quad (\Re(s) > 0, s \neq 1). \quad (1.12)$$

The Dirichlet lambda function $\lambda(s)$ is defined as the term-wise arithmetic mean of the Dirichlet eta function and the Riemann zeta function:

$$\lambda(s) = \frac{\eta(s) + \zeta(s)}{2} = \lim_{n \rightarrow \infty} O_n^{(s)} = \sum_{j=1}^{\infty} \frac{1}{(2j-1)^s} \quad (\Re(s) > 1). \quad (1.13)$$

The function $\zeta(s)$ is the standard zeta function. The following identities are noteworthy:

$$\zeta(s, 1) = \zeta(s), \quad \zeta\left(s, \frac{1}{2}\right) = 2^s \lambda(s) \quad (1.14)$$

and

$$\zeta(s, z) = \zeta(s, n+z) + \sum_{m=0}^{n-1} \frac{1}{(m+z)^s} \quad (n \in \mathbb{Z}_{\geq 1}).$$

The alternating, or skew, harmonic numbers $A_n^{(s)}$ of order s are defined by

$$A_n^{(s)} := \sum_{j=1}^n \frac{(-1)^{j+1}}{j^s} \quad (s \in \mathbb{C}, n \in \mathbb{Z}_{\geq 1}) \quad (1.15)$$

and $A_n := A_n^{(1)}$. The alternating harmonic numbers and the harmonic numbers have the following elegant relation

$$A_n^{(s)} = H_n^{(s)} - 2^{1-s} H_{[n/2]}^{(s)}. \quad (1.16)$$

The symbol, $[x]$ indicates the greatest integer less than or equal to $x \in \mathbb{R}$. The generalized alternating harmonic numbers $A_n^{(s)}(u)$ are defined by

$$A_n^{(s)}(u) := \sum_{j=1}^n \frac{(-1)^{j+1}}{(j+u)^s} \quad (s \in \mathbb{C}, u \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}, n \in \mathbb{Z}_{\geq 1}). \quad (1.17)$$

The Dirichlet beta function $\beta(z)$ is defined by

$$\beta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^z} \quad (\Re(z) > 0), \quad (1.18)$$

and $\beta(2) = G$ is the Catalan constant. Among various properties and formulas for $\beta(z)$, we recall the followings:

$$\begin{aligned} \beta(z) &= 4^{-z} \left\{ \zeta\left(z, \frac{1}{4}\right) - \zeta\left(z, \frac{3}{4}\right) \right\} \\ &= \frac{i}{2} \{ \text{Li}_z(-i) - \text{Li}_z(i) \}, \end{aligned} \quad (1.19)$$

where the polylogarithm function $\text{Li}_p(z)$ of order p is defined by, (see, e.g. [20, p. 198])

$$\text{Li}_p(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^p} \quad (|z| \leq 1; p \in \mathbb{Z}_{\geq 2}). \quad (1.20)$$

The polylogarithm

$$\text{Li}_s(-z) = -\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{z^{-1} \exp(t) + 1} dt \quad (1.21)$$

in this context, is sometimes referred to as a Fermi–Dirac integral, see Ref. [22] or [23]. From Lewin ([15, p. 299], see also, e.g. [13, pp. 30–31], [20, pp. 197–198], [12]), Jonquière’s

relation states

$$\operatorname{Li}_t(-z) + (-1)^t \operatorname{Li}_t\left(-\frac{1}{z}\right) = -2 \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(\ln z)^{t-2j}}{(t-2j)!} \eta(2j) = 2 \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(\ln z)^{t-2j}}{(t-2j)!} \operatorname{Li}_{2j}(-1) \quad (1.22)$$

where $\operatorname{Li}_t(z)$ is a polylogarithm. The relation (1.22) can also be written in terms of Bernoulli numbers, B_j so that for $t \in \mathbb{Z}_{\geq 2}$

$$\operatorname{Li}_t(-z) + (-1)^t \operatorname{Li}_t\left(-\frac{1}{z}\right) = \frac{1}{t!} \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} (-1)^j \binom{t}{2j} (1 - 2^{1-2j}) (2\pi)^j B_{2j} \ln^{t-2j}(z).$$

The relation (1.22) provides the analytic continuation of $\operatorname{Li}_s(z)$ in (1.20) outside its circle of convergence $|z| = 1$. The Bernoulli polynomials $B_j(t)$, are defined by the generating function

$$\frac{z e^{tz}}{e^z - 1} = \sum_{j \geq 0} \frac{z^j}{j!} B_j(t), \quad \text{where } |z| < 2\pi,$$

and have the representation (see, e.g. [19, Section 1.7]).

$$B_j(t) = \sum_{k=0}^j \binom{j}{k} B_k t^{j-k} = \sum_{k=0}^j \binom{j}{k} B_{t-k} t^k \quad (1.23)$$

where $B_k = B_k(0)$ are the Bernoulli numbers represented as

$$\frac{z}{e^z - 1} = \sum_{j \geq 0} \frac{z^j}{j!} B_j, \quad \text{where } |z| < 2\pi,$$

so that

$$B_{2n+1} = 0, \quad n \in \mathbb{N}.$$

The Euler polynomials, $E_n(x)$ and Euler numbers, E_n are defined by the generating function

$$\frac{2 e^{xt}}{e^t + 1} = \sum_{n \geq 0} \frac{t^n}{n!} E_n(x), \quad \text{where } |t| < \pi,$$

and

$$\frac{2 e^t}{e^{2t} + 1} = \frac{1}{\cosh(t)} = \sum_{n \geq 0} \frac{t^n}{n!} E_n, \quad \text{where } |t| < \frac{\pi}{2}, \quad (1.24)$$

or, alternatively can be explicitly written as

$$E_n(x) = \sum_{k=0}^n \frac{1}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} (x+j)^n.$$

For $z \in \mathbb{C}$ we define the general Euler number $E_n^{(m)}$ and general Euler polynomials $E_n^{(m)}(x)$ as

$$\left(\frac{2e^z}{e^{2z} + 1} \right)^m = \frac{1}{(\cosh(z))^m} = \sum_{n \geq 0} \frac{z^n}{n!} E_n^{(m)}, \quad \text{where } |z| < \frac{\pi}{2}, \quad (1.25)$$

and

$$\left(\frac{2e^{z/m}}{e^z + 1} \right)^m = \sum_{n \geq 0} \frac{z^n}{n!} E_n^{(m)}(z), \quad \text{where } |z| < \frac{\pi}{2}.$$

Some examples of general Euler numbers are

$$E_0^{(\frac{1}{4})} = 1, \quad E_1^{(\frac{1}{4})} = -\frac{1}{4}, \quad E_2^{(\frac{1}{4})} = -\frac{3}{16}, \quad E_3^{(\frac{1}{4})} = \frac{11}{64}, \quad E_4^{(\frac{1}{4})} = \frac{57}{256}.$$

The extended harmonic numbers of order $m \in \mathbb{N}$ with index $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$ are defined by (see [24])

$$H_\alpha^{(m)} := \begin{cases} \gamma + \psi(\alpha + 1) & (m = 1), \\ \zeta(m) + \frac{(-1)^{m-1}}{(m-1)!} \psi^{(m-1)}(\alpha + 1) & (m \in \mathbb{Z}_{\geq 2}). \end{cases} \quad (1.26)$$

The case $m = 1$ in (1.26) is given in (1.7). From the definition of harmonic numbers (1.6), it follows that

$$H_\alpha^{(m)} = H_{\alpha-1}^{(m)} + \frac{1}{\alpha^m} \quad (m \in \mathbb{N}, \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}). \quad (1.27)$$

The multiplication formula for polygamma functions (see, e.g. [25, p. 14]), is:

$$\psi^{(n)}(mz) = \delta_{n,0} \ln m + \frac{1}{m^{n+1}} \sum_{j=1}^m \psi^{(n)}\left(z + \frac{j-1}{m}\right) \quad (m \in \mathbb{Z}_{\geq 1}, n \in \mathbb{Z}_{\geq 0}), \quad (1.28)$$

$\delta_{n,j}$ being the Kronecker delta.

2. An integral in two parameters

The following integral is essential in the development of our results regarding the representation of skew harmonic linear Euler sums.

Theorem 2.1: *Let $t \in \mathbb{N}$ and let $-1 \leq a < 1$. Then the following integral formulas hold true:*

$$X(a, t, \infty) = \int_0^\infty \frac{x^a \ln(x) \operatorname{Li}_t(-x^2)}{1-x^2} dx \quad (2.1)$$

$$= \frac{\pi}{2^{t+3}} \sec^2\left(\frac{a\pi}{2}\right) \left(\begin{aligned} &2^{t+1} \pi \eta(t) + 2\pi \sin\left(\frac{a\pi}{2}\right) \left(\zeta\left(t, \frac{1-a}{4}\right) - \zeta\left(t, \frac{3-a}{4}\right) \right) \\ &+ t \cos\left(\frac{a\pi}{2}\right) \left(\zeta\left(t+1, \frac{1-a}{4}\right) - \zeta\left(t+1, \frac{3-a}{4}\right) \right) \end{aligned} \right) \quad (2.2)$$

where $\operatorname{Li}_t(-x^2)$ is the polylogarithm function, $\eta(t)$ is the Dirichlet eta function and $\zeta(t, \mu)$ is the generalized zeta function described in (1.11).

Proof: From (2.1), we apply the change of variable $x^2 = y$, (then rename $y = x$), so that

$$X(a, t, \infty) = \frac{1}{4} \int_0^\infty \frac{x^{\frac{a-1}{2}} \ln(x) \operatorname{Li}_t(-x)}{1-x} dx,$$

we know utilize the Fermi–Dirac integral (1.21), described in Refs [15,23] or [25] so that

$$\begin{aligned} X(a, t, \infty) &= \frac{1}{4} \int_0^\infty \frac{x^{\frac{a-1}{2}} \ln(x) \operatorname{Li}_t(-x)}{1-x} dx \\ &= -\frac{1}{4\Gamma(t)} \int_0^\infty y^{t-1} \int_0^\infty \frac{\ln(x) x^{\frac{a-1}{2}}}{(1-x)(x^{-1} \exp(y) + 1)} dx dy \end{aligned}$$

which yields

$$\begin{aligned} &= -\frac{\pi}{\Gamma(t)} \int_0^\infty \frac{y^{t-1} \exp(i\pi + \frac{y}{2}(1+a))}{(\exp(y) + 1)(1 + \exp(i\pi))^2} \\ &\quad \left(\begin{array}{c} -\pi (\exp(-\frac{y}{2}(1+a)) + \sin(\frac{a\pi}{2})) \\ + \ln(\exp(-y)) \cos(\frac{a\pi}{2}) \end{array} \right) dy \\ &= \frac{\pi 2^{1-t} \exp(i\pi)}{4(1 + \exp(i\pi))^2} \left(\begin{array}{c} 2\pi \eta(t) \\ + 2\pi \sin(\frac{a\pi}{2}) (\zeta(t, \frac{1-a}{4}) - \zeta(t, \frac{3-a}{4})) \\ + t \cos(\frac{a\pi}{2}) (\zeta(t+1, \frac{1-a}{4}) - \zeta(t+1, \frac{3-a}{4})) \end{array} \right) \\ &= \frac{\pi \exp(i\pi)}{2^{t+1}(1 + \exp(i\pi))^2} \left(\begin{array}{c} 2\pi \eta(t) \\ + 2\pi \sin(\frac{a\pi}{2}) (\zeta(t, \frac{1-a}{4}) - \zeta(t, \frac{3-a}{4})) \\ + t \cos(\frac{a\pi}{2}) (\zeta(t+1, \frac{1-a}{4}) - \zeta(t+1, \frac{3-a}{4})) \end{array} \right) \\ &= \frac{\pi}{2^{t+3}} \sec^2\left(\frac{a\pi}{2}\right) \left(\begin{array}{c} 2^{t+1} \pi \eta(t) + 2\pi \sin(\frac{a\pi}{2}) (\zeta(t, \frac{1-a}{4}) - \zeta(t, \frac{3-a}{4})) \\ + t \cos(\frac{a\pi}{2}) (\zeta(t+1, \frac{1-a}{4}) - \zeta(t+1, \frac{3-a}{4})) \end{array} \right). \end{aligned}$$

This completes the proof and (2.2) is achieved. ■

A number of special cases for the parameter are worthy of note and these are highlighted in the next Corollary.

Corollary 2.1: *Let $a = -1$, and therefore:*

$$\begin{aligned} X(-1, t, \infty) &= \int_0^\infty \frac{\ln(x) \operatorname{Li}_t(-x^2)}{x(1-x^2)} dx = \frac{1}{4} \int_0^\infty \frac{\ln(x) \operatorname{Li}_t(-x)}{x(1-x)} dx \\ &= \frac{3}{4} \zeta(2) \eta(t) + \frac{1}{8} t(t+1) \eta(t+2). \end{aligned} \tag{2.3}$$

Let $a = 0$, and therefore:

$$X(0, t, \infty) = \int_0^\infty \frac{\ln(x) \operatorname{Li}_t(-x^2)}{1-x^2} dx = \frac{3}{2} \zeta(2) \eta(t) + 2^{t-1} \pi t \beta(t+1). \tag{2.4}$$

Let $a = \frac{1}{2}$, and therefore:

$$\begin{aligned}
 & X\left(\frac{1}{2}, t, \infty\right) \\
 &= \int_0^{\infty} \frac{\sqrt{x} \ln(x) \operatorname{Li}_t(-x^2)}{1-x^2} dx = 3\zeta(2) \eta(t) \\
 &= \frac{3\zeta(2)}{2^{t+3/2}} \left(\zeta\left(t, \frac{1}{8}\right) - \zeta\left(t, \frac{5}{8}\right) \right) + \frac{\pi t}{2^{t+5/2}} \left(\zeta\left(t+1, \frac{1}{8}\right) - \zeta\left(t+1, \frac{5}{8}\right) \right). \quad (2.5)
 \end{aligned}$$

Let $a = -\frac{1}{3}$, and therefore:

$$\begin{aligned}
 X\left(-\frac{1}{3}, t, \infty\right) &= \int_0^{\infty} \frac{\ln(x) \operatorname{Li}_t(-x^2)}{x^{1/3}(1-x^2)} dx = 2\zeta(2) \eta(t) \\
 &\quad + \frac{\pi t}{2^{t+2}\sqrt{3}} \left(\zeta\left(t+1, \frac{1}{3}\right) - \zeta\left(t+1, \frac{5}{6}\right) \right) \\
 &\quad - \frac{\zeta(2)}{2^t} \left(\zeta\left(t, \frac{1}{3}\right) - \zeta\left(t, \frac{5}{6}\right) \right). \quad (2.6)
 \end{aligned}$$

Proof: Consider

$$X(-1, t, \infty) = \int_0^{\infty} \frac{\ln(x) \operatorname{Li}_t(-x^2)}{x(1-x^2)} dx = \frac{1}{4} \int_0^{\infty} \frac{\ln(x) \operatorname{Li}_t(-x)}{x(1-x)} dx.$$

The Fermi–Dirac integral yields

$$\begin{aligned}
 X(-1, t, \infty) &= \frac{1}{4} \int_0^{\infty} \frac{\ln(x) \operatorname{Li}_t(-x)}{x(1-x)} dx \\
 &= -\frac{1}{4\Gamma(t)} \int_0^{\infty} y^{t-1} \left(\int_0^{\infty} \frac{\ln(x)}{x(1-x)(1+\exp(y)x^{-1})} dx \right) dy,
 \end{aligned}$$

which reduces to a first integral

$$\begin{aligned}
 &= \frac{1}{4\Gamma(t)} \int_0^{\infty} \frac{y^{t-1} (\pi^2 + y^2)}{2 \exp(y) + 2} dy \\
 &= \frac{3}{4} \zeta(2) \eta(t) + \frac{1}{8} t(t+1) \eta(t+2).
 \end{aligned}$$

The cases of $a = 0$, $a = \frac{1}{2}$ and $a = -\frac{1}{3}$ follow directly from (2.2). ■

In the next Lemma, we will demonstrate a series representation for the integral (2.7) on the unit interval $x \in (0, 1)$.

Lemma 2.1: Let $t \in \mathbb{N}$ and let $-1 \leq a < 1$. Then the following integral formulas hold true:

$$\begin{aligned} X(a, t, 1) &= \int_0^1 \frac{x^a \ln(x) \operatorname{Li}_t(-x^2)}{1-x^2} dx \\ &= \sum_{n \geq 1} \frac{A_n^{(t)}}{(2n+1+a)^2} = \frac{1}{4} \mathbb{S}_{t,2}^{-+} \left(0, \frac{1+a}{2} \right) \end{aligned} \tag{2.7}$$

where $A_n^{(t)}$ are the skew harmonic numbers of order t .

Proof: A Taylor series expansion in the domain, $x \in (0, 1)$ gives

$$\operatorname{Li}_t(-x^2) = \sum_{n \geq 1} \frac{(-1)^n x^{2n}}{n^t}, \quad \frac{1}{1-x^2} = \sum_{n \geq 0} x^{2n}.$$

By the Cauchy product of two convergent series, then it follows that

$$\frac{x^a \ln(x) \operatorname{Li}_t(-x^2)}{1-x^2} = - \sum_{n \geq 1} A_n^{(t)} x^{2n+a} \ln(x)$$

and therefore

$$\int_0^1 \frac{x^a \ln(x) \operatorname{Li}_t(-x^2)}{1-x^2} dx = \sum_{n \geq 1} A_n^{(t)} \int_0^1 x^{2n+a} \ln(x) dx = \sum_{n \geq 1} \frac{A_n^{(t)}}{(2n+1+a)^2}.$$

In a similar fashion, it follows that

$$\begin{aligned} X(-a, t, 1) &= \int_0^1 \frac{x^{-a} \ln(x) \operatorname{Li}_t(-x^2)}{1-x^2} dx \\ &= \sum_{n \geq 1} \frac{A_n^{(t)}}{(2n+1-a)^2} = \frac{1}{4} \mathbb{S}_{t,2}^{-+} \left(0, \frac{1-a}{2} \right). \end{aligned}$$

In subsequent evaluations, we also require the following result, which may be evaluated as a standard integral.

$$\int_0^1 \frac{x^{-a} \log^j(x)}{1-x^2} dx = \frac{(-1)^j j!}{2^{j+1}} \zeta \left(j+1, \frac{1-a}{2} \right).$$

■

In the next Theorem, we establish an identity for a linear skew harmonic Euler sum of weight $(t+2)$, $t \in \mathbb{N}$ for the parameter $a \neq 0$.

Theorem 2.2: Let $t \in \mathbb{N}$, $-1 \leq a < 1$ with $a \neq 0$. Then the following formula holds true:

$$S(a, t) = \sum_{n \geq 1} A_n^{(t)} \left\{ \frac{1}{(2n+1+a)^2} + \frac{(-1)^{t+1}}{(2n+1-a)^2} \right\} \tag{2.8}$$

$$= \frac{1}{4} \mathbb{S}_{t,2}^{-+} \left(0, \frac{1+a}{2} \right) + \frac{(-1)^{t+1}}{4} \mathbb{S}_{t,2}^{-+} \left(0, \frac{1-a}{2} \right) \quad (2.9)$$

$$= \frac{1}{4} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \left\{ H_{n-\frac{1-a}{2}}^{(2)} + (-1)^{t+1} H_{n-\frac{1+a}{2}}^{(2)} \right\} \quad (2.10)$$

$$+ \frac{1}{4} \left(\zeta(2) \eta(t) + (-1)^{t+1} \zeta(2) \eta(t) \right)$$

$$= \frac{\pi}{2^{t+3}} \sec^2 \left(\frac{a\pi}{2} \right) \left(\begin{array}{c} 2^{t+1} \pi \eta(t) + 2\pi \sin \left(\frac{a\pi}{2} \right) \left(\zeta \left(t, \frac{1-a}{4} \right) - \zeta \left(t, \frac{3-a}{4} \right) \right) \\ + t \cos \left(\frac{a\pi}{2} \right) \left(\zeta \left(t+1, \frac{1-a}{4} \right) - \zeta \left(t+1, \frac{3-a}{4} \right) \right) \end{array} \right)$$

$$- \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(t+1-2j)}{2} \eta(2j) \zeta \left(t+2-2j, \frac{1-a}{2} \right). \quad (2.11)$$

Proof: Consider the following integral on the real half line $x > 0$

$$X(a, t, \infty) = \int_0^\infty \frac{x^a \ln(x) \operatorname{Li}_t(-x^2)}{1-x^2} dx.$$

Putting

$$F(a, t; x) := \frac{x^a \ln(x) \operatorname{Li}_t(-x^2)}{1-x^2}$$

it may be seen that $\lim_{x \downarrow 0} F(a, t; x)$, $\lim_{x \uparrow \infty} F(a, t; x)$ and $\lim_{x \rightarrow 1} F(a, t; x)$ exist, in fact $\lim_{x \rightarrow 1} F(a, t; x) = -\frac{1}{2} \operatorname{Li}_t(-1) = \frac{1}{2} \eta(t)$ and it maybe expressed as

$$X(a, t, \infty) = \int_0^1 F(a, t; x) dx + \int_1^\infty F(a, t; x) dx. \quad (2.12)$$

Using the transformation $xy = 1$ in the last integral in (2.12) and recovering the variable x instead of y in the resultant integral, we obtain

$$X(a, t, \infty) = \int_0^1 F(a, t; x) dx + \int_0^1 \frac{x^a \ln(x) \operatorname{Li}_t \left(-\frac{1}{x^2} \right)}{1-x^2} dx.$$

Recalling the Jonquière's formula (1.22), the last integral is expressed as

$$X(a, t, \infty) = \int_0^1 F(a, t; x) dx + (-1)^{t+1}$$

$$\int_0^1 \frac{x^{-a} \ln(x)}{1-x^2} \left(\operatorname{Li}_t(-x^2) + 2 \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(\ln(x^2))^{t-2j}}{(t-2j)!} \eta(2j) \right) dx$$

$$= \int_0^1 F(a, t; x) dx + (-1)^{t+1} \int_0^1 F(-a, t; x) dx$$

$$+ (-1)^{t+1} \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{2^{t+1-2j}}{(t-2j)!} \eta(2j) \int_0^1 \frac{x^{-a} (\ln(x))^{t+1-2j}}{1-x^2} dx.$$

Standard integration procedure gives us the result

$$\int_0^1 \frac{x^{-a} (\ln(x))^{t+1-2j}}{1-x^2} dx = (-1)^{t+1} 2^{2j-t-2} (t+1-2j)! \zeta\left(t+2-2j, \frac{1-a}{2}\right).$$

We can express the last integral in the form

$$\begin{aligned} X(a, t, \infty) &= \int_0^1 F(a, t; x) dx + (-1)^{t+1} \int_0^1 F(-a, t; x) dx \\ &\quad - \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(t+1-2j)}{2} \eta(2j) \zeta\left(t+2-2j, \frac{1-a}{2}\right). \end{aligned}$$

We now rearrange the above relation and use the results of Theorem 2.1 and Lemma 2.1 to obtain

$$\begin{aligned} S(a, t) &= \sum_{n \geq 1} A_n^{(t)} \left\{ \frac{1}{2n+1+a} + \frac{(-1)^{t+1}}{2n+1-a} \right\} = X(a, t, \infty) \\ &\quad - \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(t+1-2j)}{2} \eta(2j) \zeta\left(t+2-2j, \frac{1-a}{2}\right). \end{aligned}$$

Replacing $X(a, t, \infty)$ with (2.2), then (2.11) follows. The representation (2.10) is achieved in the following way.

$$\begin{aligned} X(a, t, 1) &= \int_0^1 \frac{x^a \ln(x) \operatorname{Li}_t(-x^2)}{1-x^2} dx = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \sum_{r \geq 0} \int_0^1 x^{2n+2r+a} \ln(x) dx \\ &= \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \sum_{r \geq 0} \frac{1}{(2n+2r+a+1)^2} \\ &= \frac{1}{4} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \psi^{(1)}\left(n + \frac{a+1}{2}\right) \\ &= \frac{1}{4} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \left(\zeta(2) - H_{n-\frac{1+a}{2}}^{(2)} \right). \end{aligned} \tag{2.13}$$

Following the same pattern we have,

$$X(-a, t, 1) = \int_0^1 \frac{x^{-a} \ln x \operatorname{Li}_t(-x^2)}{1-x^2} dx = \frac{1}{4} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \left(\zeta(2) - H_{n-\frac{1+a}{2}}^{(2)} \right)$$

and therefore

$$\begin{aligned} X(a, t, 1) + (-1)^{t+1} X(-a, t, 1) &= \frac{1}{4} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \left(\zeta(2) - H_{n-\frac{1-a}{2}}^{(2)} \right) \\ &+ \frac{(-1)^{t+1}}{4} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \left(\zeta(2) - H_{n-\frac{1+a}{2}}^{(2)} \right) \end{aligned}$$

implies the result (2.10). ■

Remark 2.1: The symmetry properties of the skew harmonic linear Euler sum, $S(a, t)$, yields: for t an even integer and $a \neq 0$

$$S(a, t) = -S(-a, t) = -4a \sum_{n \geq 1} \frac{(2n+1) A_n^{(t)}}{(2n+1+a)^2 (2n+1-a)^2},$$

for t an odd integer

$$S(a, t) = S(-a, t) = \sum_{n \geq 1} \frac{((2n+1)^2 + a^2) A_n^{(t)}}{(2n+1+a)^2 (2n+1-a)^2}.$$

Three special cases of $S(a, t)$ in (2.8) are worthy of note and are highlighted in the next Corollary.

Corollary 2.2: (1). Let $t \in \mathbb{N}$, $a = -1$. Then the following formulas hold true:

$$\begin{aligned} S(-1, t) &= \sum_{n \geq 1} A_n^{(t)} \left\{ \frac{1}{(2n)^2} + \frac{(-1)^{t+1}}{(2n+2)^2} \right\} \\ &= \frac{3}{4} \zeta(2) \eta(t) + \frac{1}{8} t(t+1) \eta(t+2) \\ &\quad - \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(t+1-2j)}{2} \eta(2j) \zeta(t+2-2j) \end{aligned} \tag{2.14}$$

where $\eta(2j)$ is the Dirichlet eta function.

(2). Let $a = 0$, t be an odd integer. Then the following formulas hold true:

$$\begin{aligned} S(0, t) &= \frac{1}{2} \mathbb{S}_{t,2}^{-+} \left(0, \frac{1}{2} \right) = 2 \sum_{n \geq 1} \frac{A_n^{(t)}}{(2n+1)^2} = \frac{3}{2} \zeta(2) \eta(t) + 2^{t-1} \pi t \beta(t+1) \\ &\quad - \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} (t+1-2j) 2^{t+1-2j} \eta(2j) \lambda(t+2-2j), \end{aligned} \tag{2.15}$$

where $\beta(t+1)$ is the Dirichlet beta function and $\lambda(t+2-2j)$ is the Dirichlet lambda function.

(3). Let $t \in \mathbb{N}$, $a = -\frac{1}{3}$. Then the following formulas hold true:

$$\begin{aligned}
 S\left(-\frac{1}{3}, t\right) &= \sum_{n \geq 1} A_n^{(t)} \left\{ \frac{1}{\left(2n + \frac{2}{3}\right)^2} + \frac{(-1)^{t+1}}{\left(2n + \frac{4}{3}\right)^2} \right\} \\
 &= \frac{1}{4} \mathbb{S}_{t,2}^{-+} \left(0, \frac{1}{3}\right) + \frac{(-1)^{t+1}}{4} \mathbb{S}_{t,2}^{-+} \left(0, \frac{2}{3}\right) \\
 &= -2\zeta(2) \eta(t) + \frac{\pi t}{2^{t+2}\sqrt{3}} \left(\zeta\left(t+1, \frac{1}{3}\right) - \zeta\left(t+1, \frac{5}{6}\right) \right) \\
 &\quad - \frac{\zeta(2)}{2^t} \left(\zeta\left(t, \frac{1}{3}\right) - \zeta\left(t, \frac{5}{6}\right) \right) \\
 &\quad - \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(t+1-2j)}{2} \eta(2j) \lambda\left(t+2-2j, \frac{2}{3}\right). \tag{2.16}
 \end{aligned}$$

Proof: The proof follows directly from (2.11). ■

In the next section, we extend the results of the previous section by considering a more general version of the integral (2.1) thereby allowing an extension of the result (2.8).

3. The general case

This section establishes a number of general skew harmonic linear Euler sum identities.

Theorem 3.1: Let $p, t \in \mathbb{N}$, $-1 \leq a < 1$ with $a \neq 0$. Then the following formulas hold true:

$$S(a, t, p) := (-1)^p (p+1)! \sum_{n \geq 1} A_n^{(t)} \left\{ \frac{1}{(2n+1+a)^{p+2}} + \frac{(-1)^{p+t+1}}{(2n+1-a)^{p+2}} \right\} \tag{3.1}$$

$$= \sum_{n \geq 1} \frac{(-1)^{p+1} (p+1)!}{2^{p+2} n^t} \left(H_{n-\frac{1-a}{2}}^{(p+2)} + (-1)^{t+1} H_{n-\frac{1+a}{2}}^{(p+2)} \right) \tag{3.2}$$

$$\begin{aligned}
 &+ \frac{(-1)^{p+1} (p+1)!}{2^{p+2}} \zeta(p+2) \eta(t) (1 + (-1)^{t+1}) \\
 &= \frac{\partial^p}{\partial a^p} (X(a, t, \infty)) \\
 &\quad - \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(t+1-2j) p!}{2^{p+1}} \eta(2j) \binom{t+1+p-2j}{p} \zeta\left(p+t+2-2j, \frac{1-a}{2}\right), \tag{3.3}
 \end{aligned}$$

where $X(a, t, \infty)$ is given by (2.2).

Proof: From Theorem 2.2 consider (2.8) and differentiate p times with respect to the parameter a so that

$$\begin{aligned} & \frac{\partial^p}{\partial a^p} (S(a, t)) \\ &= S(a, p, t) = (-1)^p (p+1)! \sum_{n \geq 1} A_n^{(t)} \left\{ \frac{1}{(2n+1+a)^{p+2}} + \frac{(-1)^{p+t+1}}{(2n+1-a)^{p+2}} \right\} \\ &= \frac{\pi}{2^{t+3}} \frac{\partial^p}{\partial a^p} \left(\sec^2 \left(\frac{a\pi}{2} \right) \left(2^{t+1} \pi \eta(t) + 2\pi \sin \left(\frac{a\pi}{2} \right) \right. \right. \\ & \quad \left. \left. \left(\zeta \left(t, \frac{1-a}{4} \right) - \zeta \left(t, \frac{3-a}{4} \right) \right) \right) \right) \\ & \quad + \frac{\pi}{2^{t+3}} \frac{\partial^p}{\partial a^p} \left(t \cos \left(\frac{a\pi}{2} \right) \left(\zeta \left(t+1, \frac{1-a}{4} \right) - \zeta \left(t+1, \frac{3-a}{4} \right) \right) \right) \\ & \quad - \frac{\partial^p}{\partial a^p} \left(\sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(t+1-2j)}{2} \eta(2j) \zeta \left(t+2-2j, \frac{1-a}{2} \right) \right). \end{aligned}$$

After some simplification and rearrangement, we obtain the identity (3.3). The representation (3.2) can be attained from the representation (2.13)

$$\begin{aligned} & \frac{\partial^p}{\partial a^p} (X(a, t, 1) + (-1)^{t+1} X(-a, t, 1)) = \frac{\partial^p}{\partial a^p} \left(\frac{1}{4} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \psi^{(1)} \left(n + \frac{1+a}{2} \right) \right) \\ & \quad + (-1)^{t+1} \frac{\partial^p}{\partial a^p} \left(\frac{1}{4} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \psi^{(1)} \left(n + \frac{1-a}{2} \right) \right) \\ &= \frac{1}{2^{p+2}} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \left(\psi^{(p+1)} \left(n + \frac{1+a}{2} \right) + (-1)^{t+1} \psi^{(p+1)} \left(n + \frac{1-a}{2} \right) \right). \end{aligned}$$

We have the Polygamma-Harmonic number relation

$$\frac{(-1)^m}{m!} \psi^{(m)}(z+1) = H_z^{(m+1)} - \zeta(m+1)$$

so that the last equality becomes

$$\begin{aligned} &= \frac{(-1)^{p+1} (p+1)!}{2^{p+2}} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \left(H_{n-\frac{1+a}{2}}^{(p+2)} + (-1)^{t+1} H_{n-\frac{1-a}{2}}^{(p+2)} \right) \\ & \quad + \frac{(-1)^{p+1} (p+1)!}{2^{p+2}} \zeta(p+2) \eta(t) (1 + (-1)^{t+1}) \end{aligned}$$

and (3.2) is attained. The harmonic numbers $H_\alpha^{(m)}$ are extended harmonic numbers of order $m \in \mathbb{N}$ with index $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$ defined by (1.26), (see [24]) and upon simplification of the above expression we obtain (3.2). ■

There are some cases of the value of the parameter a of Theorem 3.1 which are worthy of investigation and these are given in the next Corollaries. In particular, we examine the three cases of (1). $a = 1/2$, (2). $a = 0$ and $p + t$ is an odd integer. (3). $t = p + 1$, for $t \in \mathbb{N}_0$. First we give the following required result as a Lemma.

Lemma 3.1: Let $p, t \in \mathbb{N}$, $-1 \leq a < 1$, $\alpha = \frac{1}{2}$, and put.

$$W(\alpha, p, t) = \lim_{a \rightarrow \alpha} \frac{\partial^p}{\partial a^p} (X(a, t, \infty)).$$

The following identity holds true.

$$\begin{aligned} W\left(\frac{1}{2}, p, t\right) &= \lim_{a \rightarrow \frac{1}{2}} \frac{\partial^p}{\partial a^p} (X(a, t, \infty)) \\ &= \frac{1}{8} (2\pi)^{p+2} \eta(t) \sum_{r=0}^p \binom{p}{r} \left| E_r^{(\frac{1}{4})} \right| \left| E_{p-r}^{(\frac{1}{4})} \right| \\ &\quad + \frac{1}{2^{2p+t+3/2}} \sum_{r=0}^p \binom{p}{r} 2^{2r} (2\pi)^{r+2} \left| E_{r+1}^{(\frac{1}{4})} \right| (p-r)! \binom{p+t-r-1}{p-r} \\ &\quad \times \left(\zeta\left(p+t-r, \frac{1}{8}\right) - \zeta\left(p+t-r, \frac{5}{8}\right) \right) \\ &\quad + \frac{t}{2^{2p+t+7/2}} \sum_{r=0}^p \binom{p}{r} 2^{2r} (2\pi)^{r+1} \left| E_r^{(\frac{1}{4})} \right| (p-r)! \binom{p+t-r}{p-r} \\ &\quad \times \left(\zeta\left(p+t+1-r, \frac{1}{8}\right) - \zeta\left(p+t+1-r, \frac{5}{8}\right) \right), \end{aligned} \quad (3.4)$$

where $E_r^{(m)}$ are the generalized Euler numbers, defined by (1.25) and $\zeta(z, m)$ is the generalized, or Hurwitz, zeta function.

In the case of $\alpha = 0$ the following relation holds true.

$$\begin{aligned} W(0, p, t) &= \lim_{a \rightarrow 0} \frac{\partial^p}{\partial a^p} (X(a, t, \infty)) \\ &= \left(\frac{\pi}{2}\right)^{p+2} \eta(t) \sum_{r=0}^p \binom{p}{r} \left| E_r^{(\frac{1}{4})} \right| \left| E_{p-r} \right| \\ &\quad + \frac{1}{2^t} \sum_{r=0}^p \binom{p}{r} \left(\frac{\pi}{2}\right)^{r+2} \left| E_{r+1} \right| (p-r)! \binom{p+t-r-1}{p-r} \beta(p+t-r) \\ &\quad + \frac{t}{2^{t+2}} \sum_{r=0}^p \binom{p}{r} \left(\frac{\pi}{2}\right)^{r+1} \left| E_r \right| (p-r)! \binom{p+t-r}{p-r} \beta(p+t+1-r), \end{aligned} \quad (3.5)$$

where $E_r^{(m)}$ are the generalized Euler numbers and $\beta(z)$ is the Dirichlet beta function.

Proof: From (2.2) we have

$$\begin{aligned}
 & W\left(\frac{1}{2}, p, t\right) \\
 &= \lim_{a \rightarrow \frac{1}{2}} \frac{\partial^p}{\partial a^p} (X(a, t, \infty)) \\
 &= \frac{\pi}{2^{t+3}} \lim_{a \rightarrow \frac{1}{2}} \frac{\partial^p}{\partial a^p} \left(\sec^2\left(\frac{a\pi}{2}\right) \left(\begin{array}{l} 2^{t+1} \pi \eta(t) + 2\pi \sin\left(\frac{a\pi}{2}\right) \\ \times \left(\zeta\left(t, \frac{1-a}{4}\right) - \zeta\left(t, \frac{3-a}{4}\right) \right) \end{array} \right) \right) \\
 &+ \frac{\pi}{2^{t+3}} \lim_{a \rightarrow \frac{1}{2}} \frac{\partial^p}{\partial a^p} \left(t \cos\left(\frac{a\pi}{2}\right) \left(\zeta\left(t+1, \frac{1-a}{4}\right) - \zeta\left(t+1, \frac{3-a}{4}\right) \right) \right). \quad (3.6)
 \end{aligned}$$

We require the consecutive derivatives,

$$\begin{aligned}
 \lim_{a \rightarrow \frac{1}{2}} \frac{\partial^p}{\partial a^p} \left(\sec^2\left(\frac{a\pi}{2}\right) \right) &= \sum_{r=0}^p \binom{p}{r} \frac{2^{r+1} \pi^r}{\sqrt{2}} \left| E_r^{(\frac{1}{4})} \right| \frac{2^{p-r+1} \pi^{p-r}}{\sqrt{2}} \left| E_{p-r}^{(\frac{1}{4})} \right| \\
 &= 2(2\pi)^p \sum_{r=0}^p \binom{p}{r} \left| E_r^{(\frac{1}{4})} \right| \left| E_{p-r}^{(\frac{1}{4})} \right|. \quad (3.7)
 \end{aligned}$$

Next,

$$\begin{aligned}
 & \lim_{a \rightarrow \frac{1}{2}} \frac{\partial^p}{\partial a^p} \left(\sec^2\left(\frac{a\pi}{2}\right) \sin\left(\frac{a\pi}{2}\right) \left(\zeta\left(t, \frac{1-a}{4}\right) - \zeta\left(t, \frac{3-a}{4}\right) \right) \right) \\
 &= \sum_{r=0}^p \binom{p}{r} \frac{8(2\pi)^r}{2^{2p-2r}\sqrt{2}} \left| E_{r+1}^{(\frac{1}{4})} \right| (p-r)! \binom{p+t-r-1}{p-r} \\
 &\times \left(\zeta\left(p+t-r, \frac{1}{8}\right) - \zeta\left(p+t-r, \frac{5}{8}\right) \right). \quad (3.8)
 \end{aligned}$$

Further, the derivative expression

$$\begin{aligned}
 & \lim_{a \rightarrow \frac{1}{2}} \frac{\partial^p}{\partial a^p} \left(\sec\left(\frac{a\pi}{2}\right) \left(\zeta\left(t+1, \frac{1-a}{4}\right) - \zeta\left(t+1, \frac{3-a}{4}\right) \right) \right) \\
 &= \sum_{r=0}^p \binom{p}{r} \frac{2(2\pi)^r}{2^{2p-2r}\sqrt{2}} \left| E_r^{(\frac{1}{4})} \right| (p-r)! \binom{p+t-r}{p-r} \\
 &\times \left(\zeta\left(p+t+1-r, \frac{1}{8}\right) - \zeta\left(p+t+1-r, \frac{5}{8}\right) \right), \quad (3.9)
 \end{aligned}$$

substituting (3.9), (3.8) and (3.7) into (3.6) and simplifying we obtain (3.4). Similar analysis on

$$W(0, p, t) = \lim_{a \rightarrow 0} \frac{\partial^p}{\partial a^p} (X(a, t, \infty))$$

leads to the result (3.5). ■

Corollary 3.1: Let $p, t \in \mathbb{N}$, and put, $a = \frac{1}{2}$. Then the following formula holds true:

$$\begin{aligned}
 S\left(\frac{1}{2}, t, p\right) &= (-1)^p (p+1)! \sum_{n \geq 1} A_n^{(t)} \left\{ \frac{2^{p+2}}{(4n+3)^{p+2}} + \frac{2^{p+2} (-1)^{p+t+1}}{(4n+1)^{p+2}} \right\} \quad (3.10) \\
 &= \frac{(-1)^p (p+1)!}{2^{p+2}} \left(\mathbb{S}_{t,p+2}^{-+} \left(0, \frac{3}{4}\right) + (-1)^{p+t+1} \mathbb{S}_{t,p+2}^{-+} \left(0, \frac{1}{4}\right) \right) \\
 &= \frac{1}{8} (2\pi)^{p+2} \eta(t) \sum_{r=0}^p \binom{p}{r} \left| E_r^{(\frac{1}{4})} \right| \left| E_{p-r}^{(\frac{1}{4})} \right| \\
 &\quad + \frac{1}{2^{2p+t+3/2}} \sum_{r=0}^p \binom{p}{r} 2^{2r} (2\pi)^{r+2} \left| E_{r+1}^{(\frac{1}{4})} \right| (p-r)! \binom{p+t-r-1}{p-r} \\
 &\quad \times \left(\zeta\left(p+t-r, \frac{1}{8}\right) - \zeta\left(p+t-r, \frac{5}{8}\right) \right) \\
 &\quad + \frac{t}{2^{2p+t+7/2}} \sum_{r=0}^p \binom{p}{r} 2^{2r} (2\pi)^{r+1} \left| E_r^{(\frac{1}{4})} \right| (p-r)! \binom{p+t-r}{p-r} \\
 &\quad \times \left(\zeta\left(p+t+1-r, \frac{1}{8}\right) - \zeta\left(p+t+1-r, \frac{5}{8}\right) \right) \\
 &\quad - \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(t+1-2j)p!}{2^{p+1}} \eta(2j) \binom{t+1+p-2j}{p} \zeta\left(p+t+2-2j, \frac{1}{8}\right), \quad (3.11)
 \end{aligned}$$

where $E_r^{(m)}$ is the general Euler number, $\eta(z)$ is the Dirichlet eta function and $\zeta(z, \frac{1}{8})$ is the Hurwitz zeta function.

Proof: For $a = \frac{1}{2}$ and $p, t \in \mathbb{N}$ Theorem 3.1 provides:

$$\begin{aligned}
 S\left(\frac{1}{2}, t, p\right) &= (-1)^p (p+1)! \sum_{n \geq 1} A_n^{(t)} \left\{ \frac{1}{(2n+3/2)^{p+2}} + \frac{(-1)^{p+t+1}}{(2n+1/2)^{p+2}} \right\} \\
 &= \frac{(-1)^p (p+1)!}{2^{p+1}} \left(\mathbb{S}_{t,p+2}^{-+} \left(0, \frac{3}{4}\right) + (-1)^{p+t+1} \mathbb{S}_{t,p+2}^{-+} \left(0, \frac{1}{4}\right) \right) \\
 &= W\left(\frac{1}{2}, p, t\right) \\
 &\quad - \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} \frac{(t+1-2j)p!}{2^{p+1}} \eta(2j) \binom{t+1+p-2j}{p} \zeta\left(p+t+2-2j, \frac{1}{8}\right)
 \end{aligned}$$

where upon substituting $W(\frac{1}{2}, p, t)$, from (3.4) we obtain (3.11). ■

Corollary 3.2: For $\alpha = 0$, and $p + t$ is of odd weight, with $p, t \in \mathbb{N}$, the following formula is valid:

$$\begin{aligned}
 S(0, t, p) &= 2(-1)^p (p+1)! \sum_{n \geq 1} \frac{A_n^{(t)}}{(2n+1)^{p+2}} = \frac{(-1)^p (p+1)!}{2^{p+1}} \mathbb{S}_{t, p+2}^{-+} \left(0, \frac{1}{2}\right) \\
 &= \left(\frac{\pi}{2}\right)^{p+2} \eta(t) \sum_{r=0}^p \binom{p}{r} |E_r| |E_{p-r}| \\
 &\quad + \frac{1}{2^t} \sum_{r=0}^p \binom{p}{r} \left(\frac{\pi}{2}\right)^{r+2} |E_{r+1}| (p-r)! \binom{p+t-r-1}{p-r} \beta(p+t-r) \\
 &\quad + \frac{t}{2^{t+2}} \sum_{r=0}^p \binom{p}{r} \left(\frac{\pi}{2}\right)^{r+1} |E_r| (p-r)! \binom{p+t-r}{p-r} \beta(p+t+1-r) \\
 &\quad - \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} (t+1-2j) p! \eta(2j) \binom{t+1+p-2j}{p} 2^{t+1+p-2j} \lambda(t+2+p-2j),
 \end{aligned} \tag{3.12}$$

where in the last term we have used the identity from (1.14)

$$\zeta\left(t+2+p-2j, \frac{1}{2}\right) = 2^{t+2+p-2j} \lambda(t+2+p-2j),$$

where $\lambda(s)$ is the Dirichlet lambda function defined by (1.13) and $E_r^{(m)}$ is the general Euler number.

Proof: The proof follows directly from Corollary 3.12. We remark that the special case of $t = 1$ has been examined in the paper [16], ■

Another special case worthy of mention is for the situation when $t = p + 1$, this is detailed in the next Corollary.

Corollary 3.3: Let $t = p + 1 \in \mathbb{N}$. Then the following relation is valid:

$$\begin{aligned}
 S(0, p+1, p) &= 2(-1)^p (p+1)! \sum_{n \geq 1} \frac{A_n^{(p+1)}}{(2n+1)^{p+2}} = \frac{(-1)^p (p+1)!}{2^{p+1}} \mathbb{S}_{p+1, p+2}^{-+} \left(0, \frac{1}{2}\right) \\
 &= \left(\frac{\pi}{2}\right)^{p+2} \eta(t) \sum_{r=0}^p \binom{p}{r} |E_r| |E_{p-r}| \\
 &\quad + \frac{1}{2^t} \sum_{r=0}^p \binom{p}{r} \left(\frac{\pi}{2}\right)^{r+2} |E_{r+1}| (p-r)! \binom{2p-r}{p-r} \beta(2p+1-r)
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{(p+1)}{2^{p+3}} \sum_{r=0}^p \binom{p}{r} \left(\frac{\pi}{2}\right)^{r+1} |E_r| (p-r)! \binom{2p+1-r}{p-r} \beta(2p+2-r) \\
 &- \sum_{j=0}^{\lfloor \frac{p+1}{2} \rfloor} (p+2-2j) p! \eta(2j) \binom{2+2p-2j}{p} 2^{2+2p-2j} \lambda(3+2p-2j), \quad (3.13)
 \end{aligned}$$

where E_r are the Euler numbers and $\beta(z)$ is the Dirichlet beta function.

Proof: Follows directly from (3.12). ■

4. Reciprocity identity

The identity of Theorem 3.1, namely (3.1)

$$\begin{aligned}
 S(a, t, p) &= (-1)^p (p+1)! \sum_{n \geq 1} A_n^{(t)} \left\{ \frac{1}{(2n+1+a)^{p+2}} + \frac{(-1)^{p+t+1}}{(2n+1-a)^{p+2}} \right\} \\
 &= \sum_{n \geq 1} \frac{(-1)^{p+1} (p+1)! (-1)^{n+1}}{2^{p+2} n^t} \left(H_{n-\frac{1-a}{2}}^{(p+2)} + (-1)^{t+1} H_{n-\frac{1+a}{2}}^{(p+2)} \right) \quad (4.1)
 \end{aligned}$$

$$+ \frac{(-1)^{p+1} (p+1)!}{2^{p+2}} \zeta(p+2) \eta(t) (1 + (-1)^{t+1}) \quad (4.2)$$

can be thought of as satisfying a certain reciprocity (or shuffle) relation, so that using (1.2) we are able to write

$$\begin{aligned}
 &\mathbb{S}_{t,p+2}^{-+} \left(0, \frac{1+a}{2} \right) + (-1)^{p+t+1} \mathbb{S}_{t,p+2}^{-+} \left(0, \frac{1-a}{2} \right) \\
 &+ \sum_{n \geq 1} \frac{(-1)^{n+1} H_{n-\frac{1-a}{2}}^{(p+2)}}{n^t} + (-1)^{t+1} \sum_{n \geq 1} \frac{(-1)^{n+1} H_{n-\frac{1+a}{2}}^{(p+2)}}{n^t} \\
 &= - (1 + (-1)^{t+1}) \zeta(p+2) \eta(t).
 \end{aligned}$$

The case $a = 0$, delivers the following result.

Let $a = 0$ and $p, t \in \mathbb{N}$ then,

$$\sum_{n \geq 1} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^{(p+2)}}{n^t} = \zeta(p+2) \eta(t) - \mathbb{S}_{t,p+2}^{-+} \left(0, \frac{1}{2} \right), \quad (4.3)$$

where $\mathbb{S}_{t,p+2}^{-+} \left(0, \frac{1}{2} \right)$ is given by (3.12).

A related Theorem to (4.2) is enunciated by Alzer and Choi [3] regarding a general shuffle relation.

Theorem 4.1: *The following formula is given by Alzer and Choi [3], p.14. Let $p, q \in \mathbb{N}$, $\alpha, b \in \mathbb{C} \setminus \mathbb{Z}^-$, with $\alpha \neq b$, then,*

$$\mathbb{S}_{p,q}^{+-}(\alpha, b) + \mathbb{S}_{q,p}^{-+}(b, \alpha) = \eta(q, b+1) \zeta(p, \alpha+1) + \sum_{k \geq 1} \frac{(-1)^{k+1}}{(k+\alpha)^p (k+b)^q}. \quad (4.4)$$

The infinite sum can be expressed as finite linear combination of polygamma functions. Here $\eta(q, b+1)$ and $\zeta(p, \alpha+1)$ are the generalized eta function and zeta functions respectively.

We can now apply Theorem 4.1 to obtain reciprocity relations for some identities of Section 3. In the case $\alpha = 0$ and $b = \frac{1}{2}$ we have

$$\begin{aligned} \mathbb{S}_{t,p+2}^{-+}\left(0, \frac{1}{2}\right) + \mathbb{S}_{p+2,t}^{+-}\left(\frac{1}{2}, 0\right) \\ = \eta(t) \zeta\left(p+2, \frac{3}{2}\right) + \sum_{k \geq 1} \frac{(-1)^{k+1}}{k^t \left(k + \frac{1}{2}\right)^{p+2}} \end{aligned} \quad (4.5)$$

where $\eta(t)$ is the eta function, $\zeta\left(p+2, \frac{3}{2}\right)$ is the generalized zeta function and

$$\mathbb{S}_{p+2,t}^{+-}\left(\frac{1}{2}, 0\right) = \sum_{n \geq 1} \frac{(-1)^{n+1} H_n^{(p+2)}\left(\frac{1}{2}\right)}{n^t} = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^t} \sum_{j=1}^n \frac{1}{\left(j + \frac{1}{2}\right)^{p+2}}.$$

Consider the following Corollary.

Corollary 4.1: *Let $p, t \in \mathbb{N}$. Then the following formula holds true:*

$$\begin{aligned} \mathbb{S}_{p+2,t}^{+-}(0, 0, 2) = \sum_{n \geq 1} \frac{(-1)^{n+1} H_{2n}^{(p+2)}}{n^t} = \eta(t) \left(\eta(p+2) + \frac{\zeta(p+2)}{2^{p+2}} \right) \\ + \frac{1}{2^{p+2}} \left(\mathbb{S}_{p+2,t}^{+-} - \mathbb{S}_{t,p+2}^{-+}\left(0, \frac{1}{2}\right) \right), \end{aligned} \quad (4.6)$$

where $\eta(t)$ is the Dirichlet eta function and $\zeta(p+2)$ is the zeta function.

Proof: Follows directly from (4.3), (4.5) and the known relation

$$H_{n-\frac{1}{2}}^{(m)} = 2^m H_{2n}^{(m)} - 2^m \eta(m) - H_n^{(m)}.$$

■

The interesting connection between (4.3) and (4.5) gives the following remark.

Remark 4.1: From (4.3) and (4.5), we have

$$\zeta(p+2) \eta(t) - \sum_{n \geq 1} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^{(p+2)}}{n^t} + \mathbb{S}_{p+2,t}^{+-}\left(\frac{1}{2}, 0\right)$$

$$= \eta(t) \zeta\left(p+2, \frac{3}{2}\right) + \sum_{k \geq 1} \frac{(-1)^{k+1}}{k^t \left(k + \frac{1}{2}\right)^{p+2}}$$

and rearranging,

$$\begin{aligned} \mathbb{S}_{p+2,t}^{+-}\left(\frac{1}{2}, 0\right) &= \sum_{n \geq 1} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^{(p+2)}}{n^t} \\ &= \eta(t) \zeta\left(p+2, \frac{3}{2}\right) - \zeta(p+2) \eta(t) + \sum_{k \geq 1} \frac{(-1)^{k+1}}{k^t \left(k + \frac{1}{2}\right)^{p+2}}. \end{aligned}$$

Noting that $\zeta\left(p+2, \frac{3}{2}\right) = 2^{p+2}(\lambda(p+2) - 1)$ we can write that

$$\begin{aligned} \mathbb{S}_{p+2,t}^{+-}\left(\frac{1}{2}, 0\right) &= \sum_{n \geq 1} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^{(p+2)}}{n^t} \\ &= \eta(t) \left(2^{p+2}(\lambda(p+2) - 1) - \zeta(p+2)\right) + \sum_{k \geq 1} \frac{(-1)^{k+1}}{k^t \left(k + \frac{1}{2}\right)^{p+2}}. \end{aligned} \tag{4.7}$$

5. Some examples

1. Let $(a, t, p) = (0, t, 1)$, the the following identity holds, from (3.12)

$$\begin{aligned} \sum_{n \geq 1} \frac{A_n^{(t)}}{(2n+1)^3} &= \frac{1}{8} \mathbb{S}_{t,3}^{-+}\left(0, \frac{1}{2}\right) \\ &= \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} (t+1-2j)(t+2-2j) \eta(2j) 2^{t-1-2j} \lambda(t+3-2j) \\ &\quad - 2^{t-5} \pi^3 \beta(t) - 2^{t-3} \pi t(t+1) \beta(t+2). \end{aligned} \tag{5.1}$$

2. Consider (4.6) with $p = 1, t = 4$;

$$\begin{aligned} \mathbb{S}_{3,4}^{+-}(0, 0, 2) &= \sum_{n \geq 1} \frac{(-1)^{n+1} H_{2n}^{(3)}}{n^4} = \eta(4) \left(\eta(3) + \frac{\zeta(3)}{2^3} \right) \\ &\quad + \frac{1}{2^3} \left(\mathbb{S}_{3,4}^{+-} - \mathbb{S}_{4,3}^{-+}\left(0, \frac{1}{2}\right) \right). \end{aligned}$$

Using (5.1) and a standard evaluation for $\mathbb{S}_{3,4}^{+-} = \frac{363\zeta(7)}{128} - \frac{9\zeta(2)\zeta(5)}{8}$ finally reveals the result

$$\begin{aligned} \mathbb{S}_{3,4}^{+-}(0, 0, 2) &= \sum_{n \geq 1} \frac{(-1)^{n+1} H_{2n}^{(3)}}{n^4} = -\frac{121577\zeta(7)}{12024} - \frac{753\zeta(2)\zeta(5)}{64} \\ &\quad + \frac{\pi^3}{2} \beta(4) + 40\pi \beta(6). \end{aligned}$$

6. Concluding remarks

In this paper, we have considered families of the four distinct classes of linear harmonic Euler sums (1.2) and extended their representation in terms of special functions. In particular, we have developed explicit identities for the Euler sums (1.3), (1.4), (1.5) and (4.6). This work also offers ideas into further research for the development of the Euler sum representation of non-linear type, such as $\sum_{j \geq 1} \frac{H_j^{(3)}(H_j^{(4)})^2(H_j)^6}{j^r}$. A number of pertinent examples were also given to highlight the theorems and corollaries.

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