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Research Article

Iterative Refinements of the Hermite-Hadamard Inequality, Applications to the Standard Means

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Two adjacent recursive processes converging to the mean value of a real-valued convex function are given. Refinements of the Hermite-Hadamard inequality are obtained. Some applications to the special means are discussed. A brief extension for convex mappings with variables in a linear space is also provided.

1. Introduction

Let C be a nonempty convex subset of $\mathbb{R}$ and let $f : C \to \mathbb{R}$ be a convex function. For $a, b \in C$, the following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

is known in the literature as the Hermite-Hadamard inequality for convex functions. Such inequality is very useful in many mathematical contexts and contributes as a tool for establishing some interesting estimations.

In recent few years, many authors have been interested to give some refinements and extensions of the Hermite-Hadamard inequality (1.1), [1–4]. Dragomir [1] gave a refinement of the left side of (1.1) as summarized in the next result.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and let $H : [0,1] \to \mathbb{R}$ be defined by*

$$H(t) := \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx. \tag{1.2}$$

Then H is convex increasing on $[0,1]$, and for all $t \in [0,1]$, one has

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.3}$$

Yang and Hong [3] gave a refinement of the right side of (1.1) as itemized below.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and let $F : [0,1] \to \mathbb{R}$ be defined by*

$$F(t) := \frac{1}{2(b-a)}\int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\right]dx. \tag{1.4}$$

Then F is convex increasing on $[0,1]$, and for all $t \in [0,1]$, one has

$$\frac{1}{b-a}\int_a^b f(x)dx = F(0) \le F(t) \le F(1) = \frac{f(a)+f(b)}{2}. \tag{1.5}$$

From the above theorems we immediately deduce the following.

Corollary 1.3. *With the above, there holds*

$$H(t) \le \frac{1}{b-a}\int_a^b f(x)dx \le F(s), \tag{1.6}$$

for all $t, s \in [0,1]$, with

$$\inf_{0\le t\le 1} F(t) = \sup_{0\le t\le 1} H(t) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.7}$$

The following refinement of (1.1) is also well-known.

Theorem 1.4. *With the above, the following double inequality holds*

$$\begin{aligned}\left(f\left(\frac{a+b}{2}\right) \le\right)&\frac{1}{2}\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right)\\ &\le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{2}\left(f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right)\left(\le \frac{f(a)+f(b)}{2}\right).\end{aligned} \tag{1.8}$$

For the sake of completeness and in order to explain the key idea of our approach to the reader we will reproduce here the proof of the above known theorem.

Proof. Applying (1.1) successively in the subintervals $[a,(a+b)/2]$ and $[(a+b)/2,b]$ we obtain

$$f\left(\frac{3a+b}{4}\right) \le \frac{2}{b-a}\int_a^{(a+b)/2} f(x)dx \le \frac{1}{2}\left(f(a)+f\left(\frac{a+b}{2}\right)\right),$$
$$f\left(\frac{a+3b}{4}\right) \le \frac{2}{b-a}\int_{(a+b)/2}^{b} f(x)dx \le \frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+f(b)\right). \tag{1.9}$$

The desired result (1.8) follows by adding the above obtained inequalities (1.9). □

In [4] Zabandan introduced an improvement of Theorem 1.4 as recited in the following. Let (x_n) and (y_n) be the sequences defined by

$$x_n = \frac{1}{2^n}\sum_{i=1}^{2^n} f\left(a+\left(i-\frac{1}{2}\right)\frac{b-a}{2^n}\right),$$
$$y_n = \frac{1}{2^n}\left[\frac{f(a)+f(b)}{2} + \sum_{i=1}^{2^n-1} f\left(\left(1-\frac{i}{2^n}\right)a+\frac{i}{2^n}b\right)\right]. \tag{1.10}$$

Theorem 1.5. *With the above, one has the following inequalities:*

$$f\left(\frac{a+b}{2}\right) = x_0 \le \cdots \le x_n \le \frac{1}{b-a}\int_a^b f(x)dx \le y_n \le \cdots \le y_0 = \frac{f(a)+f(b)}{2} \tag{1.11}$$

with the relationship

$$\inf_{n\ge 0} y_n = \sup_{n\ge 0} x_n = \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.12}$$

Notation. Throughout this paper, and for the sake of presentation, the above expressions $H(t)$ and $F(t)$ will be denoted by $H_t(a,b)$ and $F_t(a,b)$, and the sequences $(x_n),(y_n)$ by $x_n(a,b), y_n(a,b)$, respectively. Further, the middle member of inequality (1.1), usually known by the mean value of f in $[a,b]$, will be denoted by $m_f(a,b)$, that is,

$$m_f(a,b) := \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.13}$$

2. Iterative Refinements of the Hermite-Hadamard Inequality

Let C be a nonempty convex subset of $\mathbb{R}$ and let $f : C \to \mathbb{R}$ be a convex function. As already pointed out, our fundamental goal in the present section is to give some iterative refinements of (1.1) containing those recalled in the above. We start with our general viewpoint.

2.1. General Approach

Examining the proof of Theorem 1.4 we observe that the same procedure can be again recursively applied. More precisely, let us start with the next double inequality

$$\forall a, b \in C, \quad \Phi_0(a,b) \le m_f(a,b) := \frac{1}{b-a}\int_a^b f(x)dx \le \Psi_0(a,b), \tag{2.1}$$

where $\Phi_0, \Psi_0 : C \times C \to \mathbb{R}$ are two given functions. Assume that, by the same procedure as in the proof of Theorem 1.4 we have

$$\Phi_0(a,b) \le \Phi_1(a,b) \le m_f(a,b) \le \Psi_1(a,b) \le \Psi_0(a,b), \tag{2.2}$$

with the following relationships

$$\begin{aligned} \Phi_1(a,b) &= \frac{1}{2}\Phi_0\left(a, \frac{a+b}{2}\right) + \frac{1}{2}\Phi_0\left(\frac{a+b}{2}, b\right), \\ \Psi_1(a,b) &= \frac{1}{2}\Psi_0\left(a, \frac{a+b}{2}\right) + \frac{1}{2}\Psi_0\left(\frac{a+b}{2}, b\right). \end{aligned} \tag{2.3}$$

Reiterating successively the same, we then construct two sequences, denoted by $\Phi_n(a,b)$ and $\Psi_n(a,b)$, satisfying the following inequalities:

$$\Phi_n(a,b) \le \Phi_{n+1}(a,b) \le m_f(a,b) \le \Psi_{n+1}(a,b) \le \Psi_n(a,b), \tag{2.4}$$

where $\Phi_n(a,b)$ and $\Psi_n(a,b)$ are defined by the recursive relationships

$$\begin{aligned} \Phi_{n+1}(a,b) &= \frac{1}{2}\Phi_n\left(a, \frac{a+b}{2}\right) + \frac{1}{2}\Phi_n\left(\frac{a+b}{2}, b\right), \\ \Psi_{n+1}(a,b) &= \frac{1}{2}\Psi_n\left(a, \frac{a+b}{2}\right) + \frac{1}{2}\Psi_n\left(\frac{a+b}{2}, b\right). \end{aligned} \tag{2.5}$$

The initial data $\Phi_0(a,b)$ and $\Psi_0(a,b)$, which of course depend generally of the convex function f, are for the moment upper and lower bounds of inequality (1.1), respectively, and satisfying

$$\forall a, b \in C, \quad \Phi_0(a,b) \le \Phi_1(a,b), \quad \Psi_1(a,b) \le \Psi_0(a,b). \tag{2.6}$$

Summarizing the previous approach, we may state the following results.

Theorem 2.1. *With the above, the sequence $(\Phi_n(a,b))_n$ is increasing and $(\Psi_n(a,b))_n$ is a decreasing one. Moreover, the following inequalities:*

$$\Phi_0(a,b) \le \cdots \le \Phi_n(a,b) \le m_f(a,b) \le \Psi_n(a,b) \le \cdots \le \Psi_0(a,b), \tag{2.7}$$

hold true for all $n \ge 0$.

Proof. Follows from the construction of $\Phi_n(a,b)$ and $\Psi_n(a,b)$. It is also possible to prove the same by using the above recursive relationships defining $\Phi_n(a,b)$ and $\Psi_n(a,b)$. The proof is complete. □

Corollary 2.2. *The sequences $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$ both converge and their limits are, respectively, the lower and upper bounds of $m_f(a,b)$, that is,*

$$\sup_{n\geq 0}\Phi_n(a,b) \leq m_f(a,b) \leq \inf_{n\geq 0}\Psi_n(a,b). \tag{2.8}$$

Proof. According to inequalities (2.7), the sequence $(\Phi_n(a,b))_n$ is increasing upper bounded by $\Psi_0(a,b)$ while $(\Psi_n(a,b))_n$ is decreasing lower bounded by $\Phi_0(a,b)$. It follows that $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$ both converge. Passing to the limits in inequalities (2.7) we obtain (2.8), which completes the proof. □

Now, we will observe a question arising naturally from the above study: what is the explicit form of $\Phi_n(a,b)$ (and $\Psi_n(a,b)$) in terms of n, a, b? The answer to this is given in the following result.

Theorem 2.3. *With the above, for all $n \geq 1$, there hold*

$$\begin{aligned}\Phi_n(a,b) &= \frac{1}{2^n}\sum_{i=1}^{2^n}\Phi_0\left(\frac{(2^n-i+1)a+(i-1)b}{2^n}, \frac{(2^n-i)a+ib}{2^n}\right),\\ \Psi_n(a,b) &= \frac{1}{2^n}\sum_{i=1}^{2^n}\Psi_0\left(\frac{(2^n-i+1)a+(i-1)b}{2^n}, \frac{(2^n-i)a+ib}{2^n}\right).\end{aligned} \tag{2.9}$$

Proof. Of course, it is sufficient to show the first formulae which follows from a simple induction with a manipulation on the summation indices. We omit the routine details. □

After this, we can put the following question: what are the explicit limits of the sequences $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$? Before giving an answer to this question in a special case, we may state the following examples.

Example 2.4. Of course, the first choice of $\Phi_0(a,b)$ and $\Psi_0(a,b)$ is to take the upper and lower bounds of (1.1), respectively, that is,

$$\Phi_0(a,b) = f\left(\frac{a+b}{2}\right), \qquad \Psi_0(a,b) = \frac{f(a)+f(b)}{2}. \tag{2.10}$$

With this choice, we have

$$\begin{aligned}\Phi_1(a,b) &= \frac{1}{2}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right),\\ \Psi_1(a,b) &= \frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right),\end{aligned} \tag{2.11}$$

which, respectively, correspond to the lower and upper bounds of (1.8). By convexity of f, it is easy to see that the inequalities (2.6) are satisfied. In this case we will prove in the next subsection that $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$ coincide with $(x_n(a,b))_n$ and $(y_n(a,b))_n$, respectively, and so both converge to $m_f(a,b)$.

Example 2.5. Following Corollary 1.3 we can take

$$\Phi_0(a,b) = H_t(a,b), \qquad \Psi_0(a,b) = F_s(a,b), \tag{2.12}$$

for fixed $t, s \in [0,1]$. It is not hard to verify that the inequalities (2.6) are here satisfied. In this case, our above approach defines us two sequences which depend on the variable $t \in [0,1]$. For this, such sequences of functions will be denoted by $(\Phi_{n,t})_n$ and $(\Psi_{n,t})_n$. This example, which contains the above one, will be detailed in the following.

2.2. Case of Example 2.4

Choosing $\Phi_0(a,b)$ and $\Psi_0(a,b)$ as in Example 2.4, we first state the following result.

Proposition 2.6. *With* (2.10)*, one has*

$$\begin{aligned} \Phi_n(a,b) &= x_n(a,b), \\ \Psi_n(a,b) &= y_n(a,b), \end{aligned} \tag{2.13}$$

where $x_n(a,b)$ and $y_n(a,b)$ are given by (1.10).

Proof. It is a simple verification from formulas (2.9) with (1.10). □

Now, we will reproduce to prove that the sequences $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$ both converge to $m_f(a,b)$ by adopting our technical approach. In fact, with (2.10) the sequences $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$ can be relied by a unique interesting relationship which, as we will see later, will simplify the corresponding proofs. Precisely, we may state the following result.

Proposition 2.7. *Assume that, for $a < b$, one has* (2.10)*. Then the following relation holds:*

$$\Psi_{n+1}(a,b) = \frac{1}{2}\Psi_n(a,b) + \frac{1}{2}\Phi_n(a,b). \tag{2.14}$$

Proof. It is a simple induction on n and we omit the details for the reader. □

Now we are in position to state the following result which gives an answer to the above question when $\Phi_0(a,b)$ and $\Psi_0(a,b)$ are chosen as in Example 2.4.

Theorem 2.8. *With* (2.10)*, the sequences $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$ are adjacent with the limit*

$$\lim_n \Phi_n(a,b) = \lim_n \Psi_n(a,b) = m_f(a,b), \tag{2.15}$$

and the following error-estimations hold

$$0 \le m_f(a,b) - \Phi_n(a,b) \le \Psi_n(a,b) - m_f(a,b) \le \frac{1}{2^n}\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right). \tag{2.16}$$

Proof. According to Corollary 2.2, the sequences $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$ both converge and by the relation (2.14) their limits are equal. Now, by virtue of (2.14) again we can write

$$\Psi_{n+1}(a,b) - m_f(a,b) = \frac{1}{2}\left(\Psi_n(a,b) - m_f(a,b)\right) + \frac{1}{2}\left(\Phi_n(a,b) - m_f(a,b)\right). \tag{2.17}$$

This, with the inequalities (2.7), yields

$$0 \le \Psi_{n+1}(a,b) - m_f(a,b) \le \frac{1}{2}\left(\Psi_n(a,b) - m_f(a,b)\right). \tag{2.18}$$

By a simple mathematical induction, we simultaneously obtain (2.15) and (2.16). Thus completes the proof. □

Remark 2.9. Starting from a general point of view, we have found again Theorem 1.5 under a new angle and via a technical approach. Furthermore, such approach stems its importance in what follows.

(i) As the reader can remark it, the proofs are here more simple as that of [4] for proving the monotonicity and computing the limit of the considered sequences. See [4, pages 3–5] for such comparison.

(ii) The sequences having $m_f(a,b)$ as limit are here defined by simple and recursive relationships which play interesting role in the theoretical study as in the computation context.

(iii) Some estimations improving those already stated in the literature are obtained here. In particular, inequalities (2.16) appear to be new for telling us that, in the numerical context, the convergence of $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$ to $m_f(a,b)$ is with geometric-speed.

2.3. Case of Example 2.5

As pointed out before, we can take

$$\Phi_{0,t}(a,b) = H_t(a,b), \qquad \Psi_{0,s}(a,b) = F_s(a,b), \tag{2.19}$$

for fixed $t,s \in [0,1]$. The function sequences $\Phi_{n,t}(a,b)$ and $\Psi_{n,t}(a,b)$ are defined, for all $t \in [0,1]$, by the recursive relationships

$$\begin{aligned}\Phi_{n+1,t}(a,b) &= \frac{1}{2}\Phi_{n,t}\left(a,\frac{a+b}{2}\right) + \frac{1}{2}\Phi_{n,t}\left(\frac{a+b}{2},b\right),\\ \Psi_{n+1,t}(a,b) &= \frac{1}{2}\Psi_{n,t}\left(a,\frac{a+b}{2}\right) + \frac{1}{2}\Psi_{n,t}\left(\frac{a+b}{2},b\right).\end{aligned} \tag{2.20}$$

By induction, it is not hard to see that the maps $t \mapsto \Phi_{n,t}(a,b)$ and $t \mapsto \Psi_{n,t}(a,b)$, for fixed $n \geq 0$, are convex and increasing.

Similarly to the above, we obtain the next result.

Theorem 2.10. *With* (2.19), *the following assertions are met.*

(1) *The function sequences* $(\Phi_{n,t}(a,b))_n$ *and* $(\Psi_{n,t}(a,b))_n$, *for fixed* $t \in [0,1]$, *are, respectively, monotone increasing and decreasing.*

(2) *For fixed* $n \geq 0$, *the functions* $t \mapsto \Phi_{n,t}(a,b)$ *and* $t \mapsto \Psi_{n,t}(a,b)$ *are (convex and) monotonic increasing.*

(3) *For all* $n \geq 0$ *and* $t, s \in [0,1]$, *one has*

$$\Phi_{n,t}(a,b) \leq m_f(a,b) \leq \Psi_{n,s}(a,b). \tag{2.21}$$

Proof. (1) By construction, as in the proof of Theorem 2.1.

(2) Comes from the recursive relationships defining $\Phi_{n,t}(a,b)$ and $\Psi_{n,t}(a,b)$.

(3) By construction as in the above. □

By virtue of the monotonicity of the sequences $(\Phi_{n,t}(a,b))_n$, $(\Psi_{n,t}(a,b))_n$ in a part, and that of the maps $t \mapsto \Phi_{n,t}(a,b)$, $t \mapsto \Psi_{n,t}(a,b)$ in another part, the double iterative-functional inequality (2.21) yields some improvements of refinements recalled in the above section. In particular, we immediately find the inequalities (1.3) and (1.6), respectively, by writing

$$x_n(a,b) = \Phi_{n,0}(a,b) \leq m_f(a,b) \leq \Psi_{n,1}(a,b) = y_n(a,b), \tag{2.22}$$

for all $n \geq 0$, and

$$H_t(a,b) = \Phi_{0,t}(a,b) \leq m_f(a,b) \leq \Psi_{0,s}(a,b) = F_s(a,b), \tag{2.23}$$

for all $t, s \in [0,1]$.

Open Question. As we have seen, for every $t \in [0,1]$, the sequences $(\Phi_{n,t}(a,b))_n$ and $(\Psi_{n,t}(a,b))_n$ both converge. What are their limits? To know if such convergence is uniform on $[0,1]$ is not obvious and appears also to be interesting.

3. Applications to Scalar Means

As already pointed out, this section will be devoted to display some applications of the above theoretical results. For this, we need some additional basic notions about special means.

For two nonnegative real numbers a and b, the arithmetic, geometric, harmonic, logarithmic, exponential (or identric) means of a and b are, respectively, defined by

$$\begin{gathered} \mathcal{A}(a,b) = \frac{a+b}{2}, \qquad \mathcal{G}(a,b) = \sqrt{ab}, \qquad \mathcal{H}(a,b) = \frac{2ab}{a+b}, \\ \mathcal{L}(a,b) = \frac{a-b}{\ln a - \ln b}, \quad a \neq b, \quad \mathcal{E}(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/b-a}, \quad a \neq b, \end{gathered} \tag{3.1}$$

with $\mathcal{L}(a,a) = \mathcal{E}(a,a) = a$. The following inequalities are well known in the literature

$$\mathcal{H}(a,b) \le \mathcal{G}(a,b) \le \mathcal{L}(a,b) \le \mathcal{E}(a,b) \le \mathcal{A}(a,b). \tag{3.2}$$

When a and b are given, the computations of $\mathcal{A}(a,b), \mathcal{H}(a,b)$ and $\mathcal{G}(a,b)$ are simple while that of $\mathcal{L}(a,b)$ and specially that of $\mathcal{E}(a,b)$ are not. So, approaching $\mathcal{L}(a,b)$ and $\mathcal{E}(a,b)$ by simple and practical algorithms appears to be interesting. That is the fundamental aim of what follows. In the following applications, we consider the choice (of Example 2.4),

$$\Phi_0(a,b) = f\left(\frac{a+b}{2}\right), \qquad \Psi_0(a,b) = \frac{f(a)+f(b)}{2}. \tag{3.3}$$

3.1. *Application 1: Approximation of the Logarithmic Mean*

Consider the convex function $f :]0,+\infty[\to \mathbb{R}$ defined by $f(x) = 1/x$. Preserving the same notations as in the previous section, the associate sequences $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$ correspond to the initial data

$$\Phi_0(a,b) = \frac{2}{a+b} := (\mathcal{A}(a,b))^{-1}, \qquad \Psi_0(a,b) = \frac{1/a+1/b}{2} = \frac{a+b}{2ab} = (\mathcal{H}(a,b))^{-1}. \tag{3.4}$$

Applying the above theoretical result to this particular case we immediately obtain the following result.

Theorem 3.1. *The sequences $(\Phi_n(a,b))_n$ and $(\Psi_n(a,b))_n$, corresponding to $f(x) = 1/x$, both converge to $(\mathcal{L}(a,b))^{-1}$ with the next estimation*

$$0 \le (\mathcal{L}(a,b))^{-1} - \Phi_n(a,b) \le \Psi_n(a,b) - (\mathcal{L}(a,b))^{-1} \le \frac{1}{2^n}\left(\frac{(a-b)^2}{2ab(a+b)}\right), \tag{3.5}$$

for all $n \ge 0$, and the following inequalities hold

$$\mathcal{H}(a,b) \le \cdots \le (\Psi_n(a,b))^{-1} \le \mathcal{L}(a,b) \le (\Phi_n(a,b))^{-1} \le \cdots \le \mathcal{A}(a,b). \tag{3.6}$$

The above theorem tells us that $\mathcal{L}(a,b)$ containing logarithm can be approached by an iterative algorithm involving only the elementary operations sum, product and inverse. Further, such algorithm is simple, recursive and practical for the numerical context, with a geometric-speed.

3.2. *Application 2: Approximation of the Identric Mean*

Let $f :]0,+\infty[\to \mathbb{R}$ be the convex map $f(x) = -\ln x$. Writing explicitly the corresponding iterative process $\Psi_n(a,b)$ we see that, for reason of simplicity, we may set

$$\forall n \ge 0, \quad \Theta_n(a,b) := \exp(-\Psi_n(a,b)). \tag{3.7}$$

The auxiliary sequence $(\Theta_n(a,b))_n$ is so recursively defined by

$$\Theta_0(a,b) = \sqrt{ab}, \qquad (\Theta_{n+1}(a,b))^2 = \Theta_n\left(a, \frac{a+b}{2}\right)\Theta_n\left(\frac{a+b}{2}, b\right). \tag{3.8}$$

As for $\Psi_n(a,b)$, it is easy to establish by a simple induction that

$$(\Theta_{n+1}(a,b))^2 = \Theta_n(a,b)\Theta_n^*(a,b), \tag{3.9}$$

where the dual sequence $(\Theta_n^*(a,b))_n$ is defined by a similar relationship as $(\Theta_n(a,b))_n$ with the initial data $\Theta_0^*(a,b) = (a+b)/2$. Our above approach allows us to announce the following interesting result.

Theorem 3.2. *The above sequence $(\Theta_n(a,b))_n$ converges to $\mathcal{E}(a,b)$ with the estimation*

$$\left(\frac{2\sqrt{ab}}{a+b}\right)^{1/2^n} \leq \frac{\Theta_n(a,b)}{\mathcal{E}(a,b)} \leq 1, \tag{3.10}$$

and the iterative inequalities hold

$$\sqrt{ab} = \Theta_0(a,b) \leq \cdots \leq \Theta_n(a,b) \leq \mathcal{E}(a,b) \leq \Theta_n^*(a,b) \leq \cdots \leq \Theta_0^*(a,b) = \frac{a+b}{2}. \tag{3.11}$$

Furthermore, one has

$$\Theta_n(a,b) = \left[\sqrt{ab}\prod_{i=1}^{2^n-1}\left(\left(1-\frac{i}{2^n}\right)a + \frac{i}{2^n}b\right)\right]^{1/2^n}. \tag{3.12}$$

Proof. It is immediate from the above general study. The details are left to the reader. □

Combining the inequalities of Theorems 3.1 and 3.2, with the fact that $\ln x < x$ for all $x > 0$, we simultaneously obtain the known inequalities (3.2). Further, the next result of convergence

$$\left[\prod_{i=1}^{2^n-1}\left(\left(1-\frac{i}{2^n}\right)a + \frac{i}{2^n}b\right)\right]^{1/2^n} \longrightarrow \mathcal{E}(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/b-a}, \tag{3.13}$$

when n goes to $+\infty$, is not obvious to establish directly. This proves again the interest of this work and the generality of our approach.

Remark 3.3. The identric mean $\mathcal{E}(a,b)$ having a transcendent expression is here approached by an algorithm, of algebraic type, utile for the theoretical study and simple for the numerical computation. Further as well-known, to define a non monotone operator mean, via Kubo-Ando theory [5], from the scalar case is not possible. Thus, our approach here could be the key idea for defining the identric mean involving operator and functional variables.

4. Extension for Real-Valued Function with Vector Variable

As well known, the Hermite-Hadamard inequality has an extension for real-valued convex functions with variables in a linear vector space E in the following sense: let $C \subset E$ be a nonempty convex of E and let $f : C \to \mathbb{R}$ be a convex function, then for all $x, y \in C$ there holds

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)\,dt \leq \frac{f(x)+f(y)}{2}. \tag{4.1}$$

In particular, in every linear normed space $(E, \|\cdot\|)$, we have

$$\begin{gathered}\left\|\frac{x+y}{2}\right\| \leq \int_0^1 \|(1-t)x+ty\|\,dt \leq \frac{\|x\|+\|y\|}{2},\\ \left\|\frac{x+y}{2}\right\|^2 \leq \int_0^1 \|(1-t)x+ty\|^2\,dt \leq \frac{\|x\|^2+\|y\|^2}{2}.\end{gathered} \tag{4.2}$$

In general, the computation of the middle side integrals of the above inequalities is not always possible. So, approaching such integrals by recursive and practical algorithms appears to be very interesting. Our aim in this section is to state briefly an analogue of our above approach, with its related fundamental results, for convex functions $f : C \to \mathbb{R}$. We start with the analogue of Theorem 1.4.

Theorem 4.1. *Let $f : C \to \mathbb{R}$ be a convex function. Then, for all $x, y \in C$, there holds*

$$\Phi_1(x,y) \leq \int_0^1 f((1-t)x+ty)\,dt \leq \Psi_1(x,y), \tag{4.3}$$

where $\Phi_1(x,y)$ and $\Psi_1(x,y)$ are given by

$$\begin{aligned}\Phi_1(x,y) &= \frac{1}{2}\left(f\left(\frac{3x+y}{4}\right)+f\left(\frac{x+3y}{4}\right)\right),\\ \Psi_1(x,y) &= \frac{1}{2}\left(f\left(\frac{x+y}{2}\right)+\frac{f(x)+f(y)}{2}\right).\end{aligned} \tag{4.4}$$

Proof. On making the change of variable $u = 2t$, we have

$$\int_0^{1/2} f((1-t)x+ty)\,dt = \frac{1}{2}\int_0^1 f\left((1-u)x+u\frac{x+y}{2}\right)du \tag{4.5}$$

while for the change of variable $u = 2t-1$ we have

$$\int_{1/2}^1 f((1-t)x+ty)\,dt = \frac{1}{2}\int_0^1 f\left((1-u)\frac{x+y}{2}+uy\right)du. \tag{4.6}$$

Now, applying the inequality (4.1), we have

$$
\begin{aligned}
f\left(\frac{3x+y}{4}\right) &\leq \int_0^1 f\left((1-u)x+u\frac{x+y}{2}\right)du \leq \frac{1}{2}\left[f(x)+f\left(\frac{x+y}{2}\right)\right],\\
f\left(\frac{x+3y}{4}\right) &\leq \int_0^1 f\left((1-u)\frac{x+y}{2}+uy\right)du \leq \frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f(y)\right].
\end{aligned}
\tag{4.7}
$$

If we divide both inequalities with 2 and add the obtained results we deduce the desired double inequality (4.3). □

Similarly, we set

$$
m_f(x,y) = \int_0^1 f((1-t)x+ty)dt. \tag{4.8}
$$

Now, the extension of our above study is itemized in the following statement.

Theorem 4.2. *Let C be a nonempty convex subset of a linear space E and $f : C \longrightarrow \mathbb{R}$ a convex function. For all $x, y \in C$, the sequences $(\Phi_n(x,y))_n$ and $(\Psi_n(x,y))_n$ defined by*

$$
\begin{aligned}
\Phi_{n+1}(x,y) &= \frac{1}{2}\Phi_n\left(x,\frac{x+y}{2}\right)+\frac{1}{2}\Phi_n\left(\frac{x+y}{2},y\right), \qquad \Phi_0(x,y) = f\left(\frac{x+y}{2}\right),\\
\Psi_{n+1}(x,y) &= \frac{1}{2}\Psi_n\left(x,\frac{x+y}{2}\right)+\frac{1}{2}\Psi_n\left(\frac{x+y}{2},y\right), \qquad \Psi_0(x,y) = \frac{f(x)+f(y)}{2},
\end{aligned}
\tag{4.9}
$$

are, respectively, monotonic increasing and decreasing and both converge to $m_f(x,y)$ with the following estimation

$$
0 \leq m_f(x,y)-\Phi_n(x,y) \leq \Psi_n(x,y)-m_f(x,y) \leq \frac{1}{2^n}\left(\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right). \tag{4.10}
$$

Proof. Similar to that of real variables. We omit the details here. □

Of course, the sequences $(\Phi_n(x,y))_n$ and $(\Psi_n(x,y))_n$ are relied by similar relation as (2.14) and explicitly given by analogue expressions of (2.9). In particular, we may state the following.

Example 4.3. Let $p \geq 1$ be a real number and let $f : E \longrightarrow \mathbb{R}$ be the convex function defined by $f(x) = \|x\|^p$. In this case, $\Phi_n(x,y)$ and $\Psi_n(x,y)$ are given by

$$
\begin{aligned}
\Phi_n(x,y) &= \frac{1}{2^{n(p+1)+p}}\sum_{i=1}^{2^n}\left\|\left(2^{n+1}-2i+1\right)x+(2i-1)y\right\|^p,\\
\Psi_n(x,y) &= \frac{1}{2^{n(p+1)+1}}\sum_{i=1}^{2^n}\left(\left\|(2^n-i+1)x+(i-1)y\right\|^p+\left\|(2^n-i)x+iy\right\|^p\right).
\end{aligned}
\tag{4.11}
$$

with the following inequalities:

$$0 \le \int_0^1 \|(1-t)x + ty\|^p dt - \Phi_n(x,y) \le \Psi_n(x,y) - \int_0^1 \|(1-t)x + ty\|^p dt \le \frac{1}{2^n}\left(\frac{\|x\|^p + \|y\|^p}{2} - \left\|\frac{x+y}{2}\right\|^p\right). \tag{4.12}$$

Remark 4.4. The Hermite-Hadamard inequality, together with some associate refinements, can be extended for nonreal-valued maps that are convex with respect to a given (partial) ordering. In this direction, we indicate the recent paper [6].

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